Pseudo Boolean Constraints :-

$x_1 \ldots x_n$ Boolean variables.

Pseudo Boolean Constraint :

$$c_1 x_1 + c_2 x_2 + \cdots + c_n x_n \leq c$$

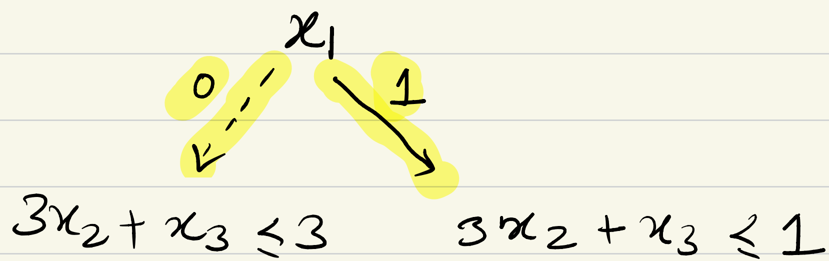
where $c_1, \ldots c_n \in \mathbb{Z}$

$$2x_1 + 3x_2 + x_3 \leq 3$$

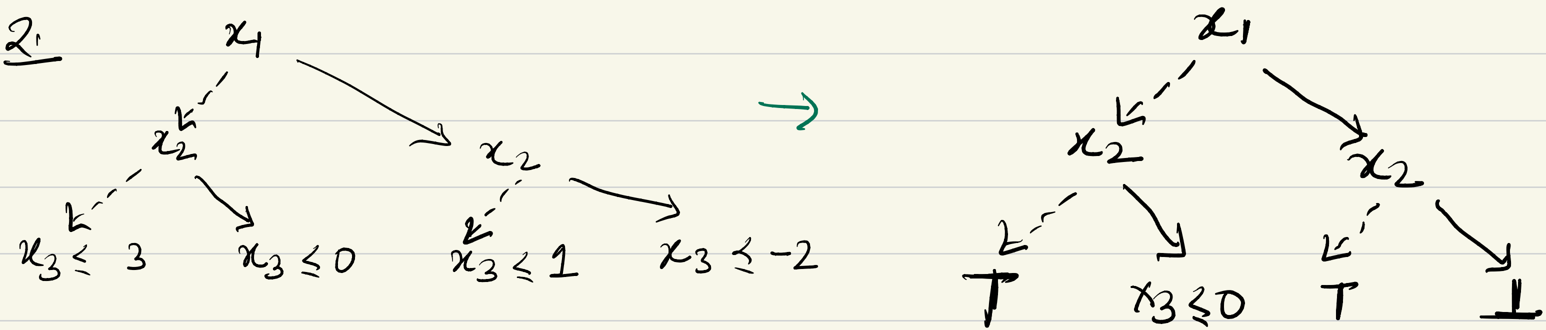
Q Solve this using Boolean reasoning.

$$2x_1 + 3x_2 + x_3 \leq 3$$

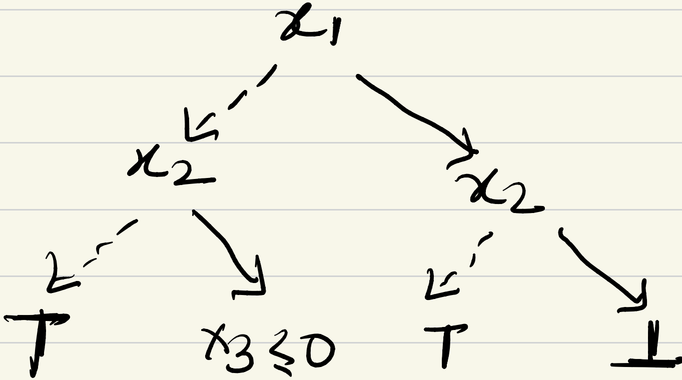
1.

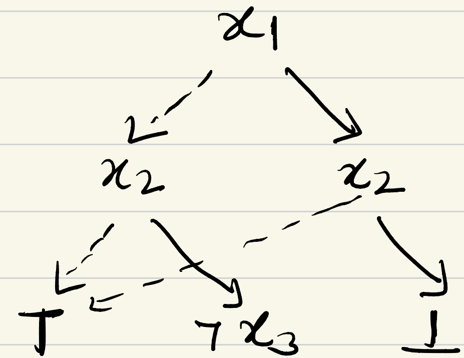


2.



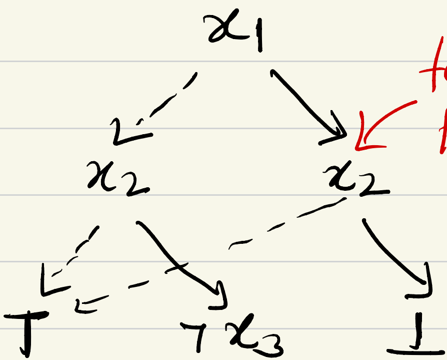
$$2x_1 + 3x_2 + x_3 \le 3$$





Reduced Ordered Binary decision diagram

$$2x_1 + 3x_2 + x_3 \le 3$$



temp variable for each internal node

$$(\neg x_1 \rightarrow temp1) \wedge$$
$$(temp1 \wedge \neg x_2 \rightarrow \top) \wedge$$
$$(temp1 \wedge x_2 \rightarrow \neg x_3) \wedge$$

$$(x_1 \rightarrow temp2) \wedge$$
$$(temp2 \wedge \neg x_2 \rightarrow \top) \wedge$$
$$(temp2 \wedge x_2 \rightarrow \bot)$$

Simplifications

* Trivially True ($\top$) ; if $c \geq c_1 + \dots + c_n$,
  then $c_1 x_1 + \dots + c_n x_n \leq c$ is $\top$

* Trivially false ($\perp$) ; if $c < 0$, then
  $c_1 x_1 + \dots + c_n x_n \leq c$ is $\perp$

* Trim large coefficients to $c+1$, let us suppose $c_i > c$

  $t + c_i x_i \leq c$ can be written as

  $t + (c+1) x_i \leq c$

* Replacing negative coefficients to positive

  $t_i - c_i x_i \leq c$ can be simplified to

  $t_i + c_i (\neg x_i) \leq c + c_i$

* Divide the whole constraints by $\gcd(c_1 \cdots c_n)$

Home work :

1. $2x_1 + 6x_2 + x_3 \leq 3$

2. $2x_1 + 3x_2 + 5x_3 \geq 6$

x_1, x_2, x_3 are Boolean variables. Convert the pseudo-Boolean inequalities into ROBDDs & thereafter into equisatisfiable formulas.

Pigeon hole Principle

Thm: If we place $n+1$ pigeons in n holes then there is a hole with at least 2 pigeons.

Thm is True for any n, but we can prove it for a fixed n.

Come with a CNF encoding for Pigeon hole Principle

3 pigeons, 2 holes

$P_{11}, P_{12}, P_{21}, P_{22}, P_{31}, P_{32}$ [P_{ij} i^{th} pigeon in j^{th} hole]

* $P_{11} \vee P_{12}$
$P_{21} \vee P_{22}$
$P_{31} \vee P_{32}$
} each pigeon sits in at least one hole

* $\neg P_{11} \vee \neg P_{21}$ $\neg P_{12} \vee \neg P_{22}$
$\neg P_{11} \vee \neg P_{31}$ $\neg P_{12} \vee \neg P_{32}$
$\neg P_{21} \vee \neg P_{31}$ $\neg P_{22} \vee \neg P_{32}$
} There is at most one pigeon in each hole.

P_{ij} :- $i \in 1..n+1$ & $j \in 1..n$

P_{ij} is 1 iff pigeon i sits in hole j

Each pigeon sits in at least one hole :-

for each $i \in 1..n$ $(P_{i1} \vee \dots \vee P_{in})$

There is at most one pigeon in each hole

$(\neg P_{ik} \vee \neg P_{jk})$ for each $k \in 1..n$

$i < j \in 1..n$

Encoding of Bayesian Network to CNF :-

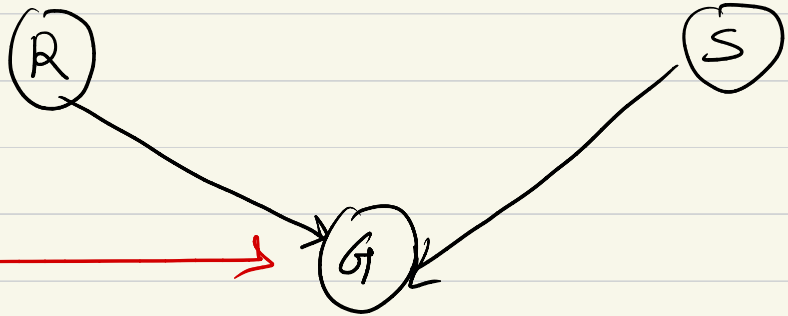
Bayesian Network :-

"Bayesian Network is a probabilistic graphical model that represents a set of variables & their conditional dependencies via a DAG (Directed Acyclic Graph)."

Conditional Probability Table

Rain	
1	0
.8	.2

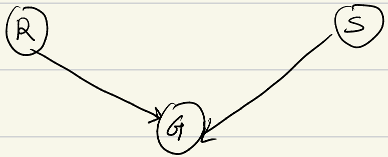
Sprinkler	
T	F
0.6	0.4



G is dependent on R & S

Rain	Sprinkler	Grass Wet T	Grass Wet F
T	T	0.9	0.1
T	F	0.8	0.2
F	T	0.7	0.3
F	F	0.4	0.6

Rain	
1	0
.8	.2



Sprinkler	
T	F
0.6	0.4

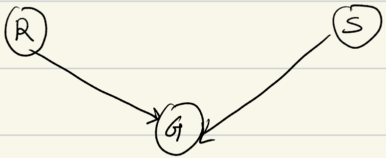
Rain	Sprinkler	Grass Wet T	Grass Wet F
T	T	0.9	0.1
T	F	0.8	0.2
F	T	0.7	0.3
F	F	0.4	0.6

Event e as R = T, S = T, G = T

then

$P(e) = ?$

Rain	
1	0
.8	.2



Sprinkler	
T	F
0.6	0.4

Rain	Sprinkler	Grass Wet T	Grass Wet F
T	T	0.9	0.1
T	F	0.8	0.2
F	T	0.7	0.3
F	F	0.4	0.6

Event e as R=T, S=T, G=T
then

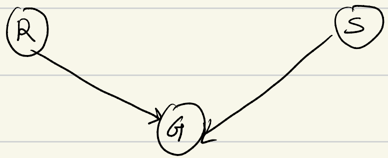
$P(e) = ?$

$$P(e) = P(R=1) \times P(S=1) \times P(G=1 \mid R=1, S=1)$$

$$= 0.8 \times 0.6 \times 0.9$$

$$= 0.432$$

Rain	
1	0
.8	.2



Sprinkler	
T	F
0.6	0.4

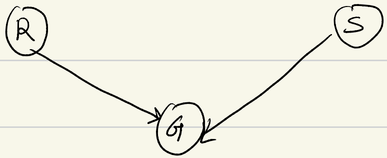
Rain	Sprinkler	Grass Wet T	Grass Wet F
T	T	0.9	0.1
T	F	0.8	0.2
F	T	0.7	0.3
F	F	0.4	0.6

Event e as $R = T$, $G = T$

then

$P(e) = ?$

Rain	
1	0
.8	.2



Sprinkler	
T	F
0.6	0.4

Event e as $R = T,\ G = T$.

then

$$P(e) = ?$$

Rain	Sprinkler	Grass Wet T	Grass Wet F
T	T	0.9	0.1
T	F	0.8	0.2
F	T	0.7	0.3
F	F	0.4	0.6

$$P(e) = P(R=T) \times P(S=T) \times P(G=T \mid R=T, S=T)$$
$$+\ P(R=T) \times P(S=F) \times P(G=T \mid R=T, S=F)$$

Q: Can we encode Bayesian Network into CNF formula F, such that probability of any event e is $W(F \wedge e)$?

Q: Can we encode Bayesian Network into CNF formula F, such that probability of any event e is $W(F \wedge e)$?

Imp Question is, how can we have a notion of probability ?

$$W : 2^{|variables(F)|} \mapsto [0,1]$$

Weight function in Propositional logic

F is defined over X variables

$$X = \{x_1, x_2, \ldots, x_n\}$$

$$W : (x_i) \mapsto [0,1]$$

$$W(x_1, \neg x_1, x_2, \neg x_2, \ldots, x_n, \neg x_n) \mapsto [0,1]$$

Special case : $W(x_i) + W(\neg x_i) = 1$

- $\tau \Vdash F$, τ is a satisfying assignment
- $W(\tau) = \prod_{x_i \in \tau} W(x_i) \cdot \prod_{x_i \notin \tau} W(\neg x_i)$
- $W(F) = \sum_{\tau \Vdash F} W(\tau)$.

$\rightarrow F = x_1 \lor x_2$

$\rightarrow W(x_1) = 0.9$, $W(\neg x_1) = 0.1$, $W(x_2) = 0.7$

$W(\neg x_2) = 0.1$

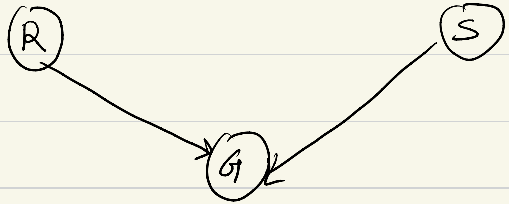
$\tau = \langle x_1 \mapsto 1, x_2 \mapsto 0 \rangle$

$W(\tau) = W(x_1) \times W(\neg x_2)$

$= 0.9 \times 0.1$

$W(\tau) \Rightarrow 0.09$

Rain	
1	0
.8	.2



Sprinkler	
T	F
0.6	0.4

Rain	Sprinkler	Grass Wet T	Grass Wet F
T	T	0.9	0.1
T	F	0.8	0.2
F	T	0.7	0.3
F	F	0.4	0.6

Q: Can we encode Bayesian Network into CNF formula F, such that probability of any event e is $W(F \wedge e)$?

Bayesian Network $\longrightarrow$ F_{CNF} & weight function.

Let us assume that I_R, I_S, I_G represent the indicator variables for Rain, Sprinkler & Grass

$e = \langle Rain = T,\ Sprinkler = T,\ Grass = False \rangle$

probability of event is $W(F \wedge I_R \wedge I_S \wedge \neg I_G)$

Indicators : I_R, I_S, I_G

parameters : P_R, P_S, P_{RSG}, $P_{\bar{R}SG}$, $P_{R\bar{S}G}$, $P_{\bar{R}\bar{S}G}$

Weight Function

$$w(x) = P(x) \quad \& \quad W(\neg x) = 1 - P(x)$$

$$x \in \{\text{Parameters}\}$$

$$w(I) = 1 \quad \& \quad w(\neg I) = 1$$

$$I \in \{\text{Indicators}\}$$

$$I_R \longleftrightarrow P_R$$
$$I_S \longleftrightarrow P_S$$

$$I_R \wedge I_S \wedge I_G \rightarrow P_{RSG}$$
$$I_R \wedge I_S \wedge \neg I_G \rightarrow \neg P_{RSG}$$

$$\neg I_R \wedge I_S \wedge I_G \rightarrow P_{\bar{R}SG}$$
$$\neg I_R \wedge I_S \wedge \neg I_G \rightarrow \neg P_{\bar{R}SG}$$

$$I_R \wedge \neg I_S \wedge I_G \rightarrow P_{R\bar{S}G}$$
$$I_R \wedge \neg I_S \wedge \neg I_G \rightarrow \neg P_{R\bar{S}G}$$

$$\neg I_R \wedge \neg I_S \wedge I_G \rightarrow P_{\bar{R}\bar{S}G}$$
$$\neg I_R \wedge \neg I_S \wedge \neg I_G \rightarrow \neg P_{\bar{R}\bar{S}G}$$

$I_R \longleftrightarrow P_R$

$I_S \longleftrightarrow P_S$

$I_R \wedge I_S \wedge I_G \rightarrow P_{RSG}$

$I_R \wedge I_S \wedge \neg I_G \rightarrow \neg P_{RSG}$

$\neg I_R \wedge I_S \wedge I_G \rightarrow P_{\bar{R}SG}$

$\neg I_R \wedge I_S \wedge \neg I_G \rightarrow \neg P_{\bar{R}SG}$

$I_R \wedge \neg I_S \wedge I_G \rightarrow P_{R\bar{S}G}$

$I_R \wedge \neg I_S \wedge \neg I_G \rightarrow \neg P_{R\bar{S}G}$

$\neg I_R \wedge \neg I_S \wedge I_G \rightarrow P_{\bar{R}\bar{S}G}$

$\neg I_R \wedge \neg I_S \wedge \neg I_G \rightarrow \neg P_{\bar{R}\bar{S}G}$

<u>Weight Function</u>

$W(x) = P(x)$ & $W(\neg x) = 1 - P(x)$

$x \in \{\text{Parameters}\}$

$w(I) = 1$ & $w(\neg I) = 1$

$I \in \{\text{Indicators}\}$

$I_R \longleftrightarrow P_R$
$I_S \longleftrightarrow P_S$

$I_R \wedge I_S \wedge I_G \rightarrow P_{RSG}$
$I_R \wedge I_S \wedge \neg I_G \rightarrow \neg P_{RSG}$

$\neg I_R \wedge I_S \wedge I_G \rightarrow P_{\bar{R}SG}$
$\neg I_R \wedge I_S \wedge \neg I_G \rightarrow \neg P_{\bar{R}SG}$

$I_R \wedge \neg I_S \wedge I_G \rightarrow P_{R\bar{S}G}$
$I_R \wedge \neg I_S \wedge \neg I_G \rightarrow \neg P_{R\bar{S}G}$

$\neg I_R \wedge \neg I_S \wedge I_G \rightarrow P_{\bar{R}\bar{S}G}$
$\neg I_R \wedge \neg I_S \wedge \neg I_G \rightarrow \neg P_{\bar{R}\bar{S}G}$

<u>Weight Function</u>

$w(x) = P(x)$ & $W(\neg x) = 1 - P(x)$

$x \in \{\text{Parameters}\}$

$w(I) = 1$ & $w(\neg I) = 1$

$I \in \{\text{Indicators}\}$

Q: $e \langle Rain = T, \; Sprinkler = T, \; Grass = T \rangle$

does $W(F \wedge I_R \wedge I_S \wedge I_G) = P(e)$?

$I_R \leftrightarrow P_R$

$I_S \leftrightarrow P_S$

$I_R \wedge I_S \wedge I_G \rightarrow P_{RSG}$

$I_R \wedge I_S \wedge \neg I_G \rightarrow \neg P_{RSG}$

$\neg I_R \wedge I_S \wedge I_G \rightarrow P_{\bar{R}SG}$

$\neg I_R \wedge I_S \wedge \neg I_G \rightarrow \neg P_{\bar{R}SG}$

$I_R \wedge \neg I_S \wedge I_G \rightarrow P_{R\bar{S}G}$

$I_R \wedge \neg I_S \wedge \neg I_G \rightarrow \neg P_{R\bar{S}G}$

$\neg I_R \wedge \neg I_S \wedge I_G \rightarrow P_{\bar{R}\bar{S}G}$

$\neg I_R \wedge \neg I_S \wedge \neg I_G \rightarrow \neg P_{\bar{R}\bar{S}G}$

Weight Function

$w(x) = P(x)$ & $W(\neg x) = 1 - P(x)$

$x \in \{\text{Parameters}\}$

$w(I) = 1$ & $w(\neg I) = 1$

$I \in \{\text{Indicators}\}$

$$W(\underbrace{F \wedge I_R \wedge I_S \wedge I_G}_{F'}) = \sum_{C \models F'} W(C)$$

$C_1 = \langle I_R, I_S, I_G, P_R, P_S, P_{RSG}, \neg P_{\bar{R}SG}, \neg P_{R\bar{S}G}, \neg P_{\bar{R}\bar{S}G} \rangle$

$C_2 = \langle I_R, I_S, I_G, P_R, P_S, P_{RSG}, P_{\bar{R}SG}, \neg P_{R\bar{S}G}, \neg P_{\bar{R}\bar{S}G} \rangle$

$C_3 = \langle I_R, I_S, I_G, P_R, P_S, P_{RSG}, P_{\bar{R}SG}, P_{RSG}, \neg P_{\bar{R}\bar{S}G} \rangle$

$$W(\underbrace{F \wedge I_R \wedge I_S \wedge I_G}_{F'}) = \sum_{\tau \Vdash F'} W(\tau)$$

$$= W(I_R) \times W(I_S) \times W(I_G) \times W(P_R) \times W(P_S) \times W(P_{RSG}) \times (1)$$

$$= 1 \times 1 \times 1 \times 0.8 \times 0.2 \times 0.9 \times .1$$

$$= \underline{\underline{0.432}}$$

Q: e $\langle$ ~~R~~ Rain = T, Grass = T $\rangle$

does $W(F \wedge I_R \wedge I_G) = P(e)$?

Q An institute is offering m courses.
↳ each has a number of contact hours == credits
The institute has r rooms
↳ each room has a maximum student capacity
The institute has s weekly slots to conduct the courses.
↳ each slot has either 1 or 1.5 hour length
There are n students:
- each student have to take minimum # of credits
- each student has a set of preferred courses.

Assign each course slot & a room such that all students can take courses from their preferred courses that meet their minimum credit criteria.
Write an encoding into SAT problem that finds such an assignment.

* Quiz - Next week.

* Project 1 → SAT Solver (out next week)
  - deadline - 27/09/2024 (11:59 pm)

* Project 2 - topic selection by 04/10/2024 (11:59pm)