COL:750

Foundations of Automatic Verification

Instructor: Priyanka Golia

Course Webpage

https://priyanka-golia.github.io/teaching/COL-750/index.html

Intro to SMT: Satisfiability Modulo Theory

FOL: grammar for a rational abstract thinking

FOL: Doesn't have a knowledge of any specific matter.

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A theory T is a set of sentences closed under implications

If $T \rightarrow F$, then $F \in T$

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We write T -satisfiability as $M \models_T F$.

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No, it is unsatisfiable, $\not\models T_{\mathbb{N}} \cup F$

If formula is satisfiable, gives an satisfying assignment

Formulas in different theories —> SMT

(Linear integer arithmetic,

Linear real arithmetic, bit vectors, strings)

Unsatisfiable

Chaff SAT Solver — 2000 (DPLL + conflict analysis, heuristics)

Order of magnitude faster than previous SAT solvers

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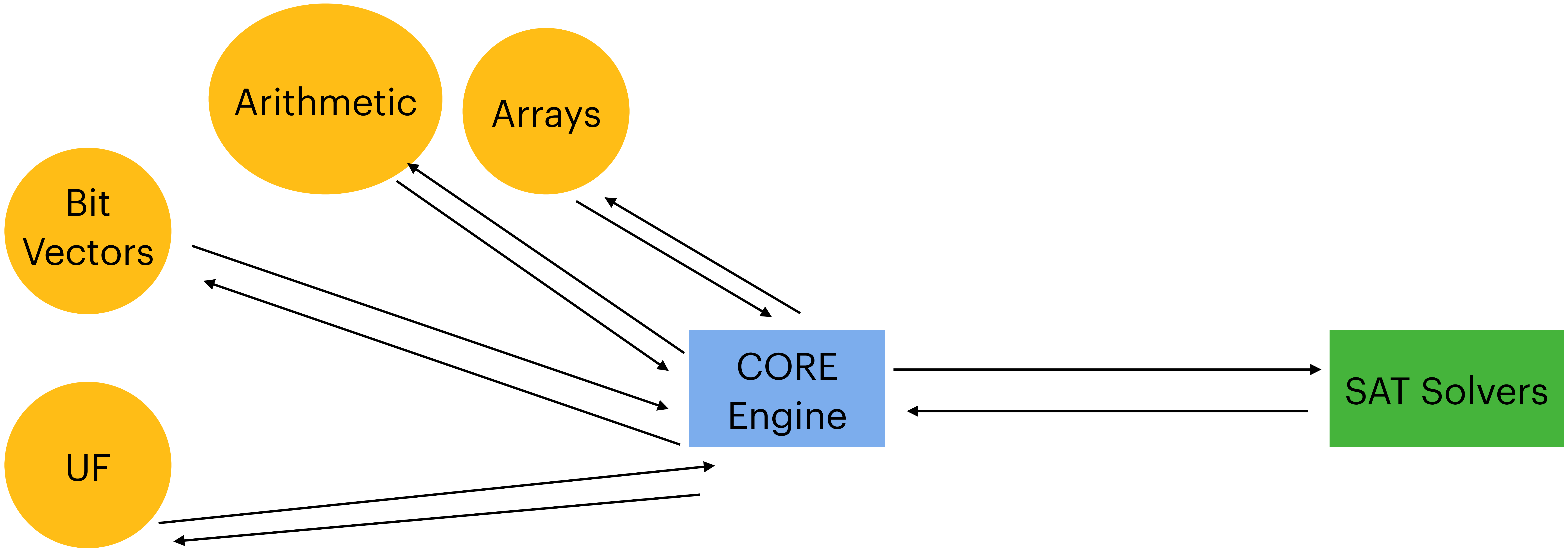
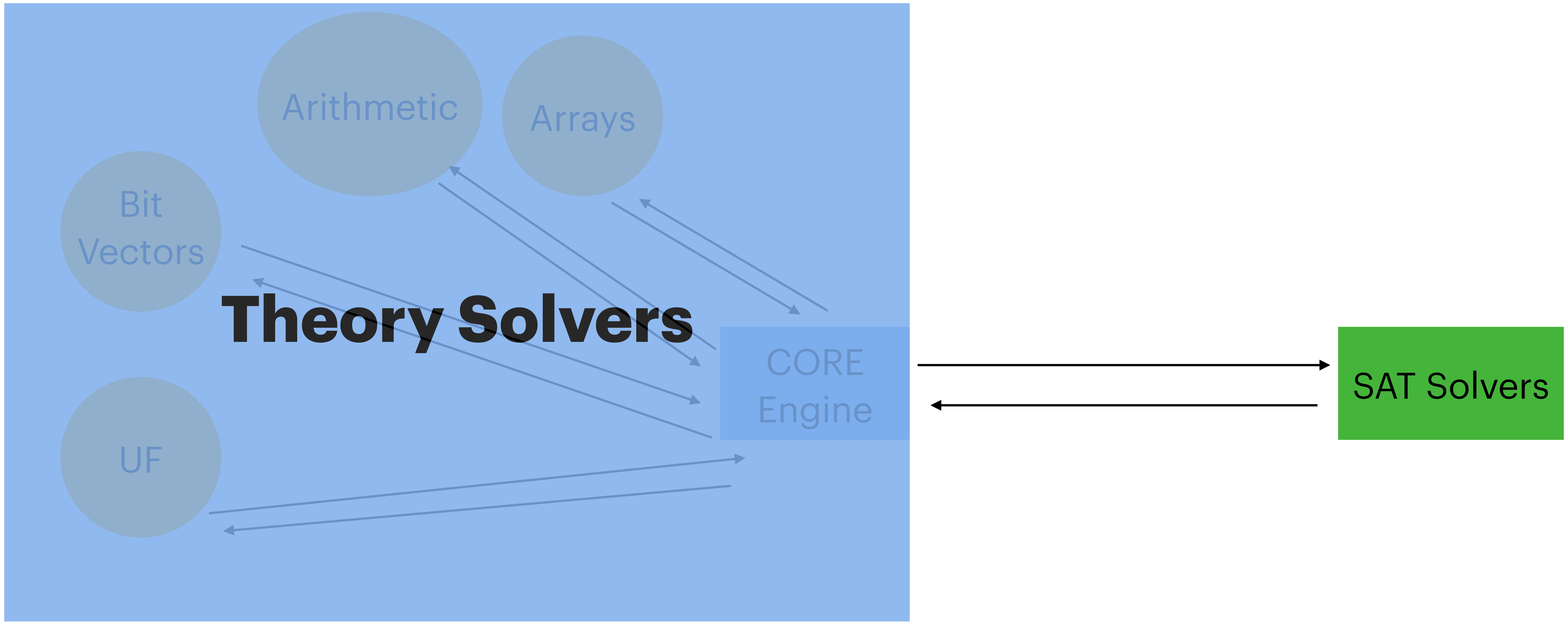
Alto, 2001, came up with idea of combining SAT solvers with decision procedures for decidable first-order theories.

SVC, CVC, Yices solver came to picture — first SMT solver was born!!!



SMT solvers


Arithmetic
Arrays
Bit Vectors
Theory Solvers
CORE Engine
UF
SAT Solvers

SMT solvers

Theory Solvers

Theory Solver: Difference Logic

Difference logic — the satisfiability of a conjunction of arithmetic atoms.

Each atom is of the form $x - y \oplus c$,
where x and y are variables, c is a numeric constant, and
$\oplus \in \{ <, >, \leq, \geq, = \}$

The variables can range over either the integers (QF_IDL) or the reals (QF_RDL).

Theory Solver: Difference Logic

The first step is to rewrite everything in terms of $\leq$

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$$x - y < c \quad \equiv x - y \leq c - 1$$ For integers

$$\equiv x - y \leq c - \delta$$ For reals

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$$x - y < c \quad \equiv x - y \leq c - 1$$ For integers

$$\equiv x - y \leq c - \delta$$ For reals

$$x - y > c \quad \equiv y - x < -c$$

Theory Solver: Difference Logic

- A conjunction of literals, all of the form $x - y \leq c$.
- From these literals, we form a weighted directed graph with a vertex for each variable.
- For each literal $x - y \leq c$, there is an edge $x \rightarrow y$, with weight c.

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$$(x - y = 5) \wedge (z - y \geq 2) \wedge (z - x > 2) \wedge (w - x = 2) \wedge (z - w < 0)$$

Theory Solvers

Linear Arithmetic Solver

Handles inequalities and equalities over integers or real numbers:

Techniques: Fourier-Motzkin elimination, Simplex algorithm.

Check if $(x + 2y \leq 10) \wedge (x - y \geq 3)$?

Bit-Vector Solver

Deals with fixed-width integers and bitwise operations:

Techniques: Bit-blasting (reducing bit-vector problems to SAT), word-level reasoning

Check if $x > > 4 = 0x0A$

Theory Solvers

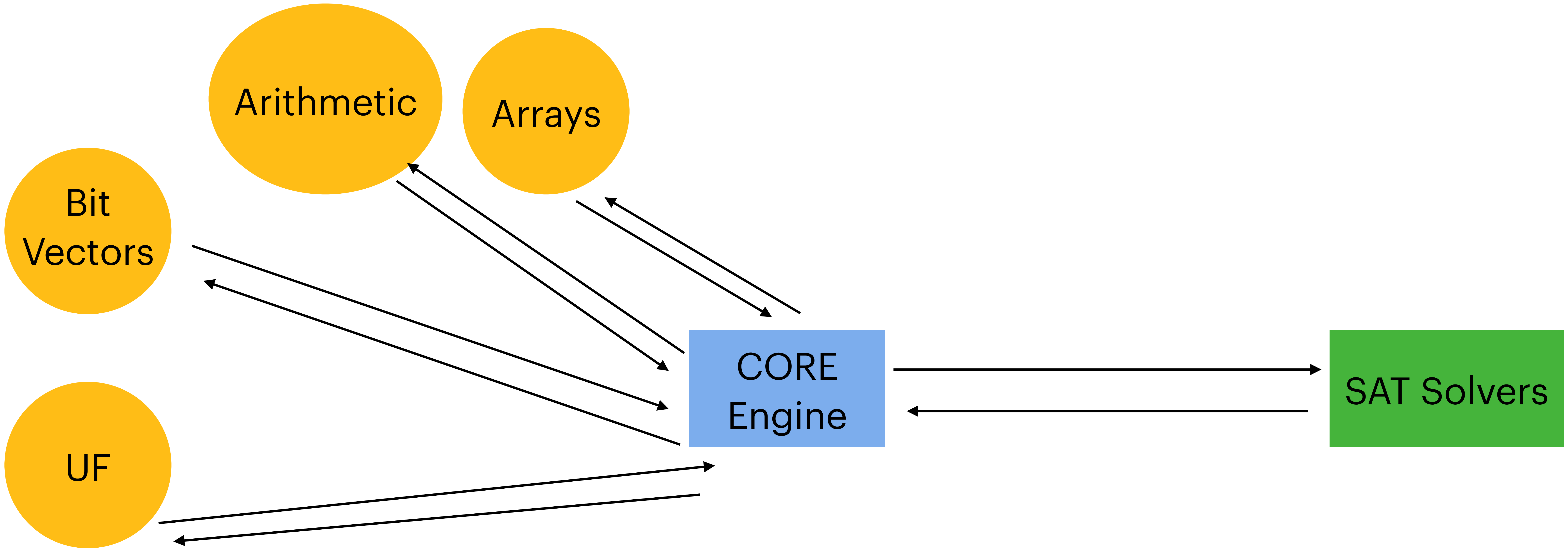
Theory Propagation

Deducing new constraints or facts based on existing ones.
For example, in linear integer arithmetic:

given $(x \geq 5) \wedge (y = x + 2)$, we can deduce $y >= 7$

Theory Consistency Checking

Check if a set of constraints is consistent within the theory.
If not, it provides a conflict (a minimal subset of constraints that are unsatisfiable)



SMT solvers

SMT Solvers

Two main approaches:

1. "Eager" approach
    1. Translate into an equisatisfiable propositional formula
    2. Feed it to any SAT solver
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    1. Abstract the input formula to a propositional formula
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   Cvc5, z3, MathSAT, OpenSMT

SMT solving — Lazy Approach

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$$(g(a) = c) \land (f(g(a)) \neq f(c) \lor g(a) = d) \land (c \neq d)$$

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$$\underbrace{(g(a) = c)}_{p_1} \wedge (\underbrace{f(g(a)) \neq f(c)}_{\neg p_2} \vee \underbrace{g(a) = d}_{p_3}) \wedge \underbrace{(c \neq d)}_{\neg p_4}$$

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Send $(p_1 \wedge (\neg p_2 \vee p_3) \wedge \neg p_4)$ to a SAT solver.

SAT solver returns $\sigma = \{p_1 \mapsto 1, p_2 \mapsto 0, p_3 \mapsto 0, p_4 \mapsto 0\}$

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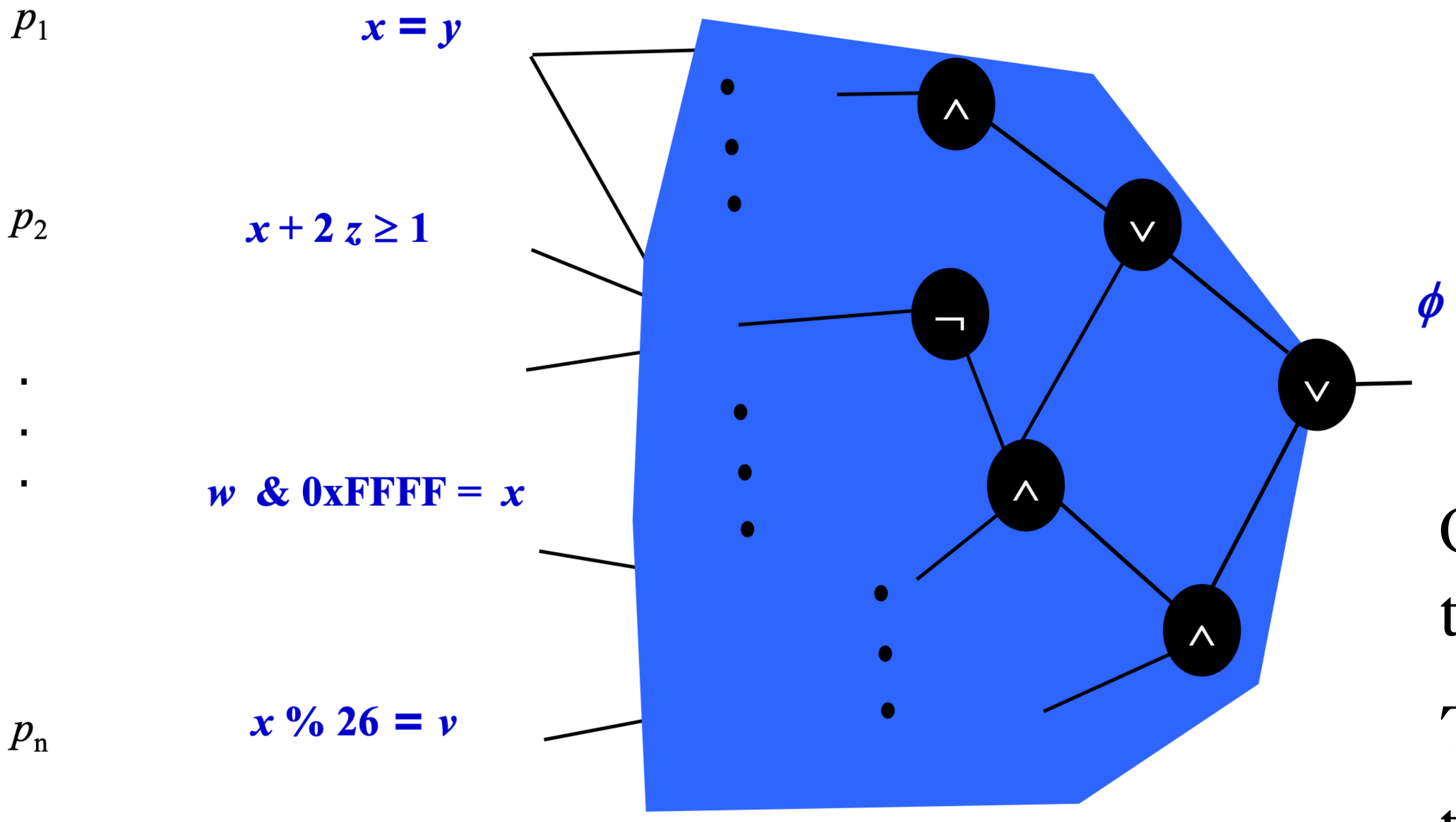
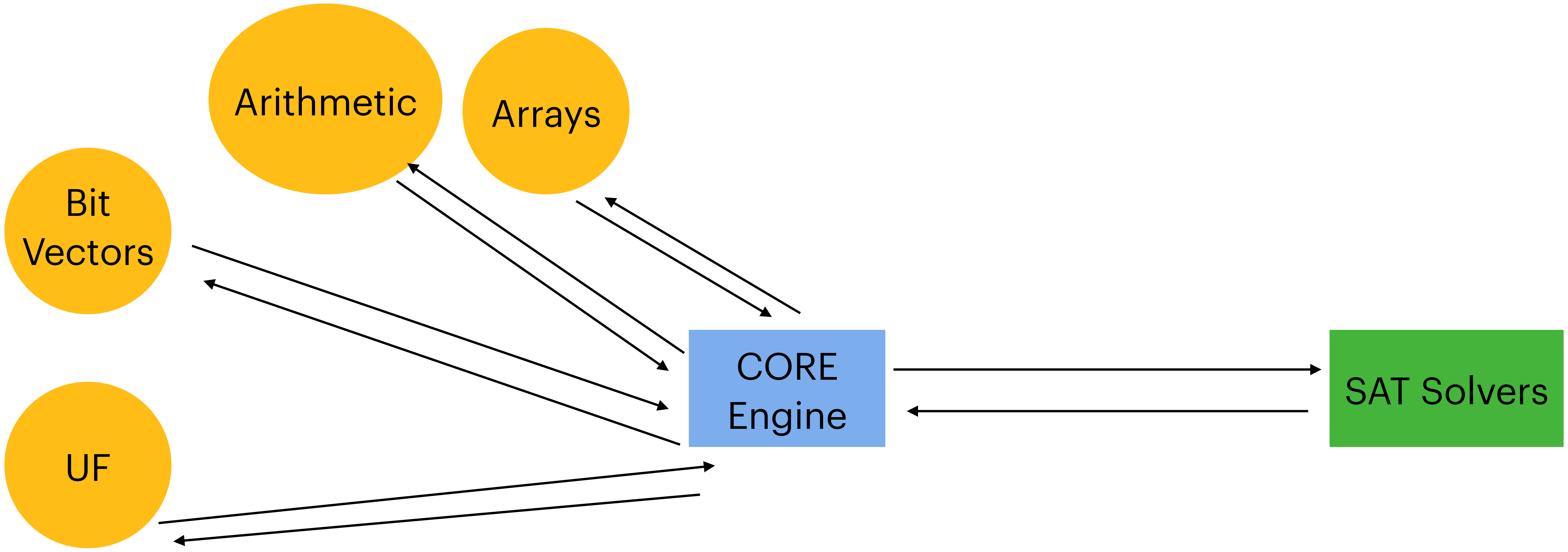
Backtrack to a point where M was still T-Satisfiable,
use this to pass more explanation to SAT solver.



Can have combinations of theories!

Task is to find an assignment to $Vars(\phi)$ such that ϕ is satisfiable!

SMT solvers



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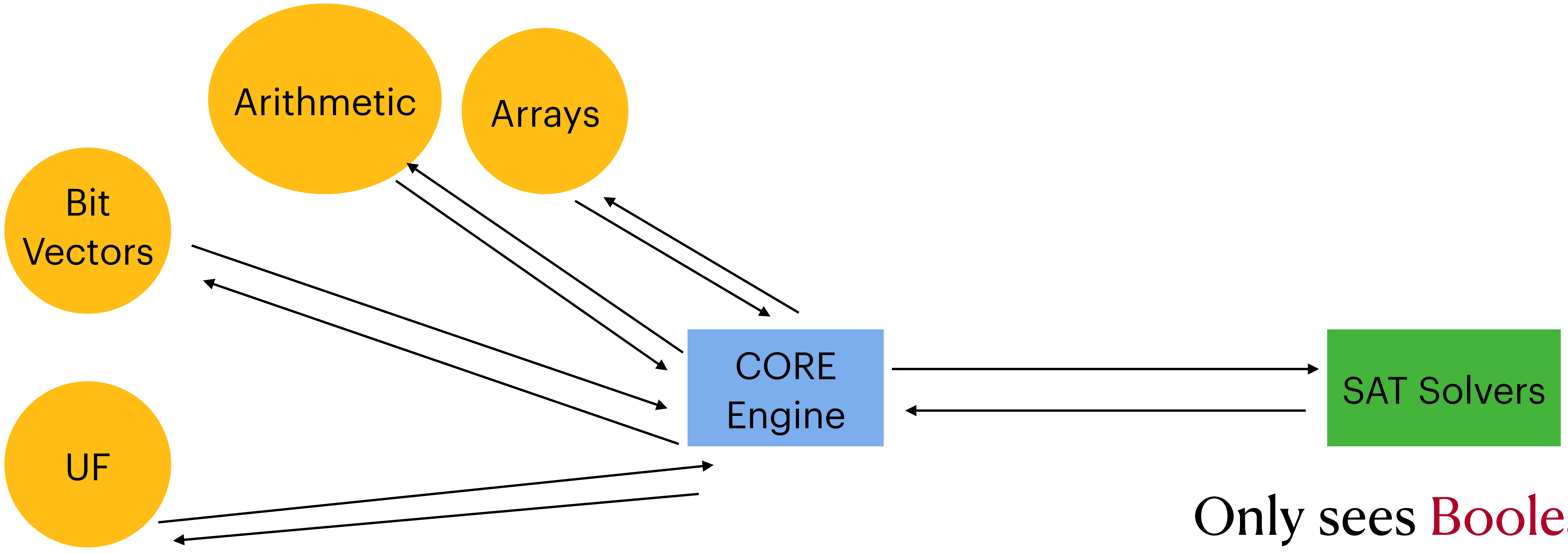


Only sees Boolean Skeleton of the problem!

Builds partial model by assigning truth values to literals

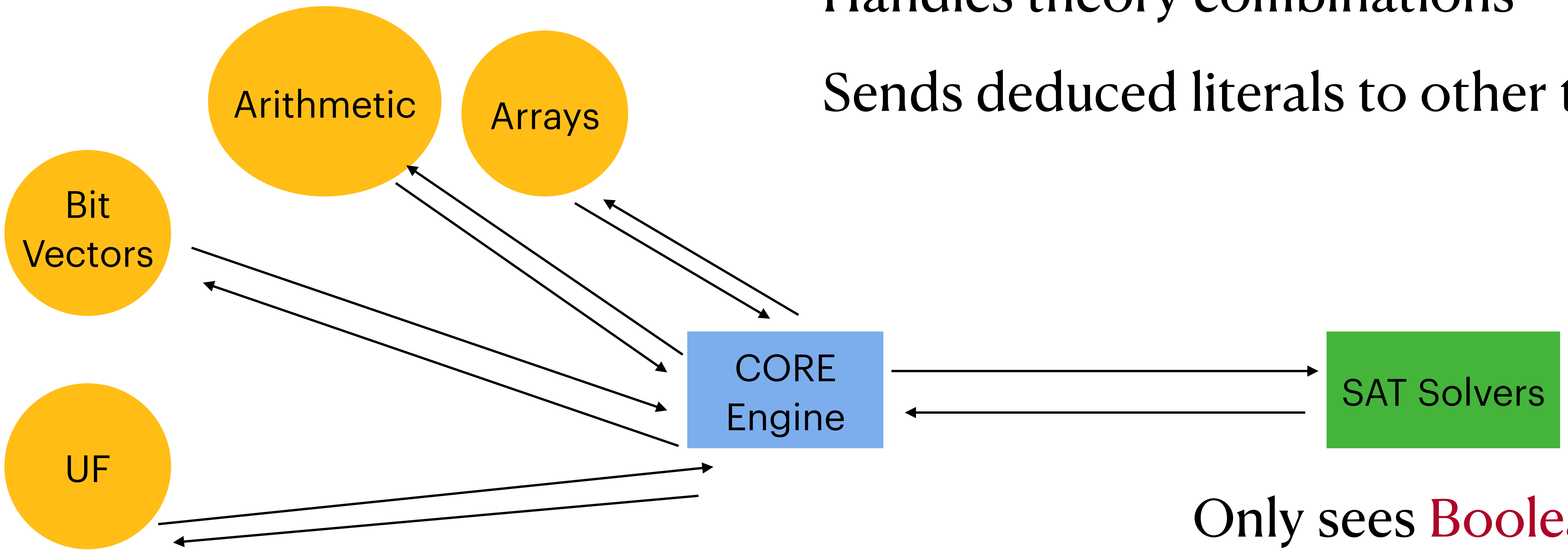
Sends these literals to the core as assertions

SMT solvers

Sends each assertions to the appropriate theory

Handles theory combinations

Sends deduced literals to other theories/SAT solver



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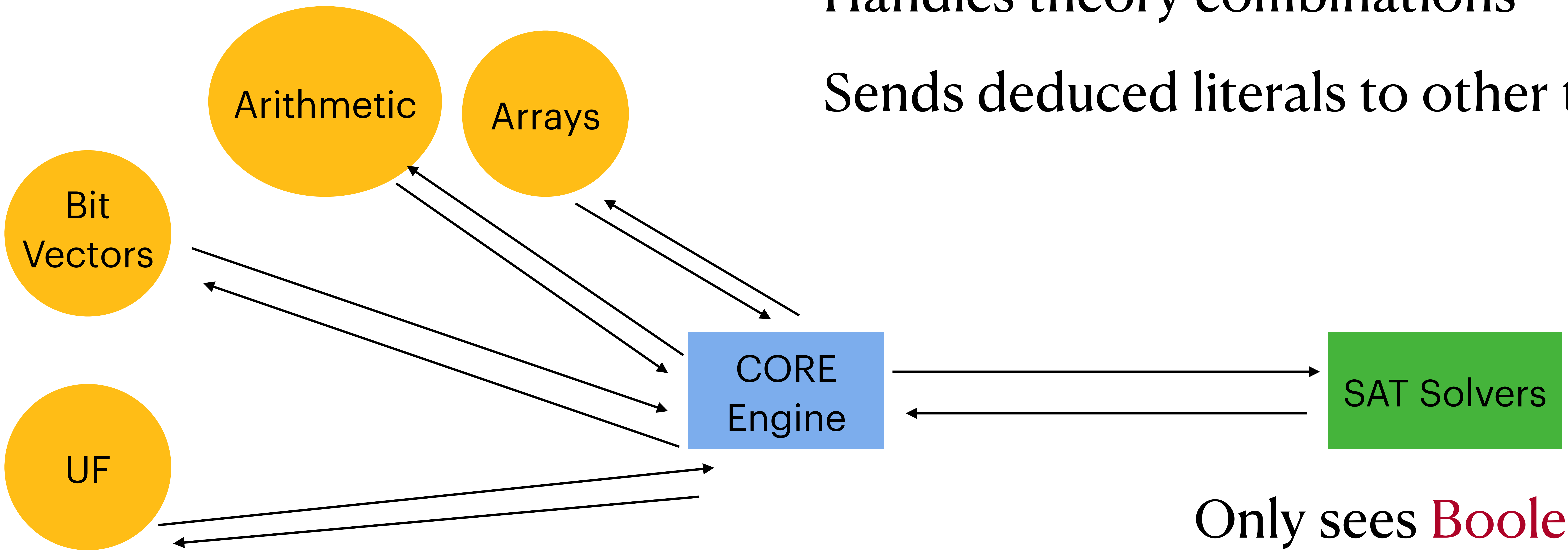

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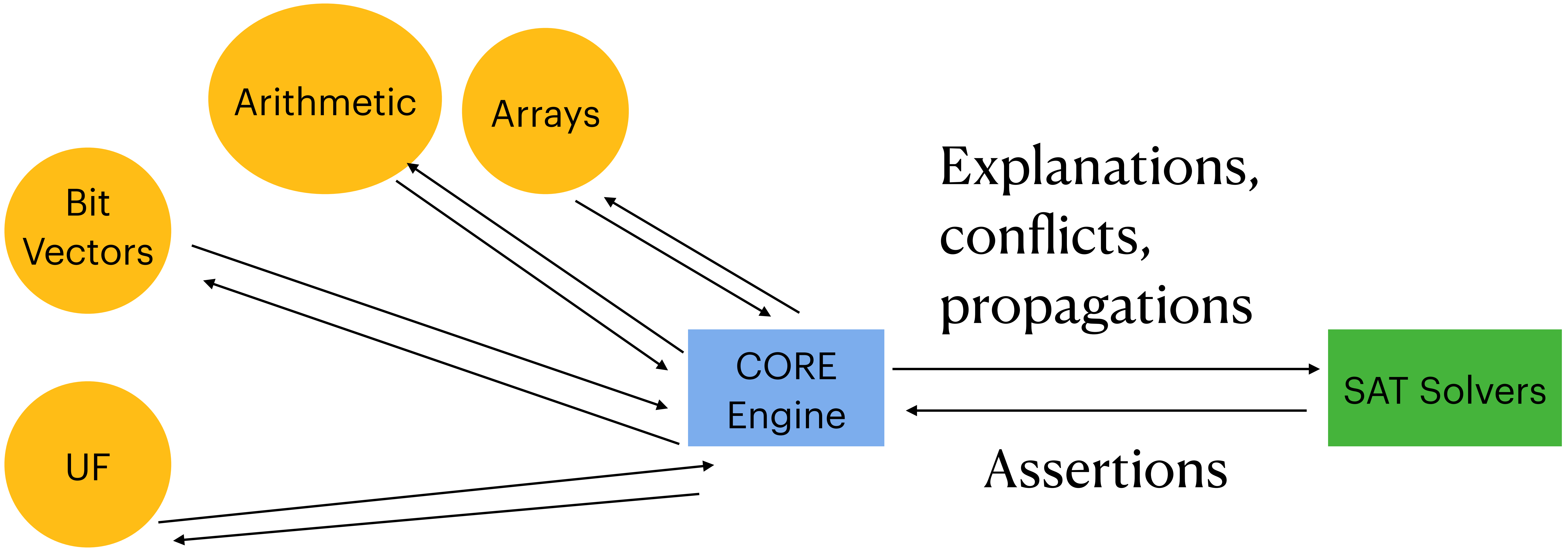
Theory Solvers!
Decide T-satisfiability of a conjunction of literals.

Builds partial model by assigning truth values to literals

Sends these literals to the core as assertions



Arithmetic
Arrays
Bit Vectors
UF
CORE Engine
Explanations, conflicts, propagations
SAT Solvers
Assertions

SMT solvers

From SAT & SMT to Temporal Logic

SAT: Checks whether a propositional formula is satisfiable.

SMT: Extends SAT with richer theories (e.g., arithmetic, arrays).

But What About Time?

SAT/SMT/FOL verify properties in static systems.

Many real-world systems evolve over time (e.g., software, robots, protocols).

"A robot should always eventually return to its charging station."
"A user who enters a correct password will eventually get access."
"How can we verify that a system never reaches an error state?"

Can we express this in SAT or FOL?

From SAT & SMT to Temporal Logic

Classical logic (SAT/SMT) = Static Reasoning

Temporal logic = Reasoning over time

Linear Temporal Logic (LTL) Assumes a single timeline (one possible sequence of events).

Next Class: Linear Temporal Logic (LTL)

Course Webpage

Thanks!