COL:750

Foundations of Automatic Verification

Instructor: Priyanka Golia

Course Webpage

https://priyanka-golia.github.io/teaching/COL-750/index.html

Boolean /propositional formulas ——> SAT Solvers

If formula is SATisfiable, gives an satisfying assignment

UNSAT

DP algorithm for SAT Solving (Martin Davis - Hilary Putnam 1960)

1. Start with F_{CNF}
2. For every clause C in F_{CNF} that either contains both l and $\neg l$ or has pure literal do:
   1. $F_{CNF} \leftarrow remove\_from\_formula(C, F_{CNF})$
3. If F_{CNF} is empty
   1. Return SAT
4. If F_{CNF} has empty clause then
   1. Return UNSAT
5. Pick a literal l that occurs with both polarities in F_{CNF}.
   1. $F_{CNF} \leftarrow Resolution(C, l, F_{CNF})$
6. For every clause C that contains l or $\neg l$ do :
   1. $F_{CNF} \leftarrow remove\_from\_formula(C, F_{CNF})$

DP algorithm

$$F = (p \lor q) \land (p \lor \neg q) \land (\neg p \lor r) \land (\neg p \lor \neg r)$$

DP algorithm

$$F = (p \lor q) \land (p \lor \neg q) \land (\neg p \lor r) \land (\neg p \lor \neg r)$$

* No pure literal, no clause with $l \lor \neg l$

DP algorithm

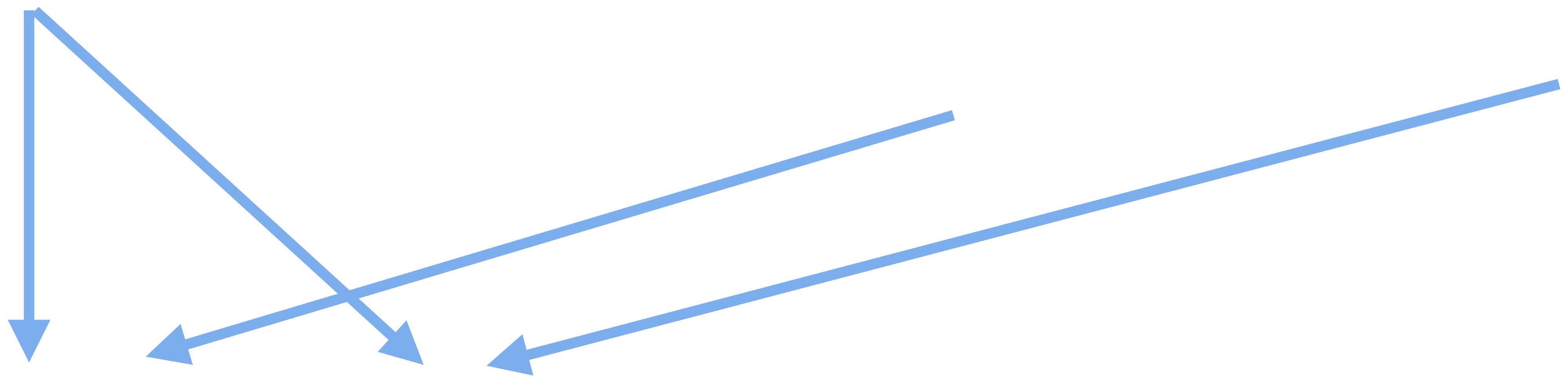
$$F = (p \lor q) \land (p \lor \neg q) \land (\neg p \lor r) \land (\neg p \lor \neg r)$$

* No pure literal, no clause with $l \lor \neg l$

Pick literal p

DP algorithm

$$F = (p \vee q) \wedge (p \vee \neg q) \wedge (\neg p \vee r) \wedge (\neg p \vee \neg r)$$



* No pure literal, no clause with $l \vee \neg l$

Pick literal p

DP algorithm

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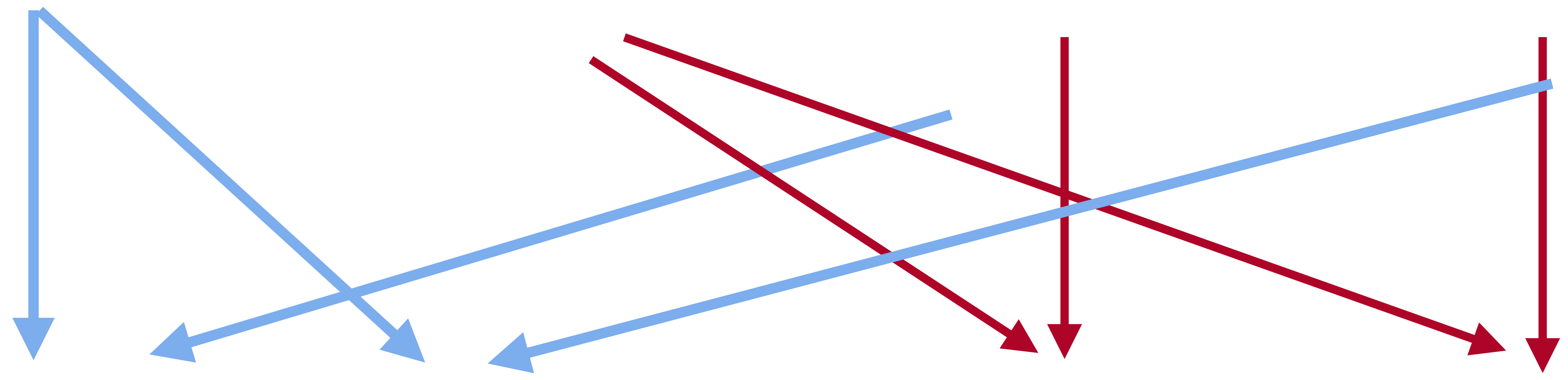

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Pick literal p

DP algorithm

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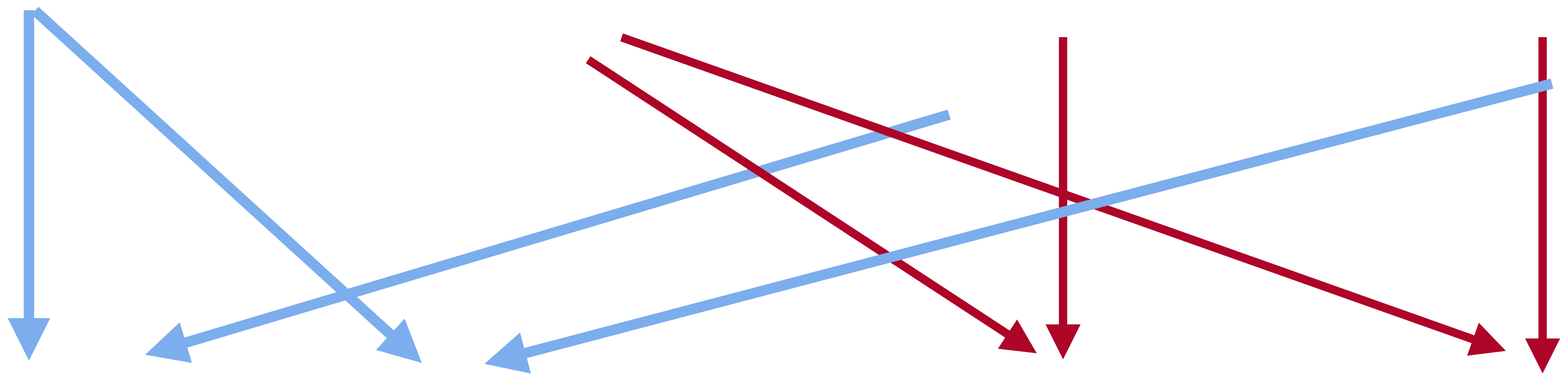


$$(q \vee r) \;\wedge\; (q \vee \neg r) \;\wedge\; (\neg q \vee r) \;\wedge\; (\neg q \vee \neg r)$$

* No pure literal, no clause with $l \vee \neg l$

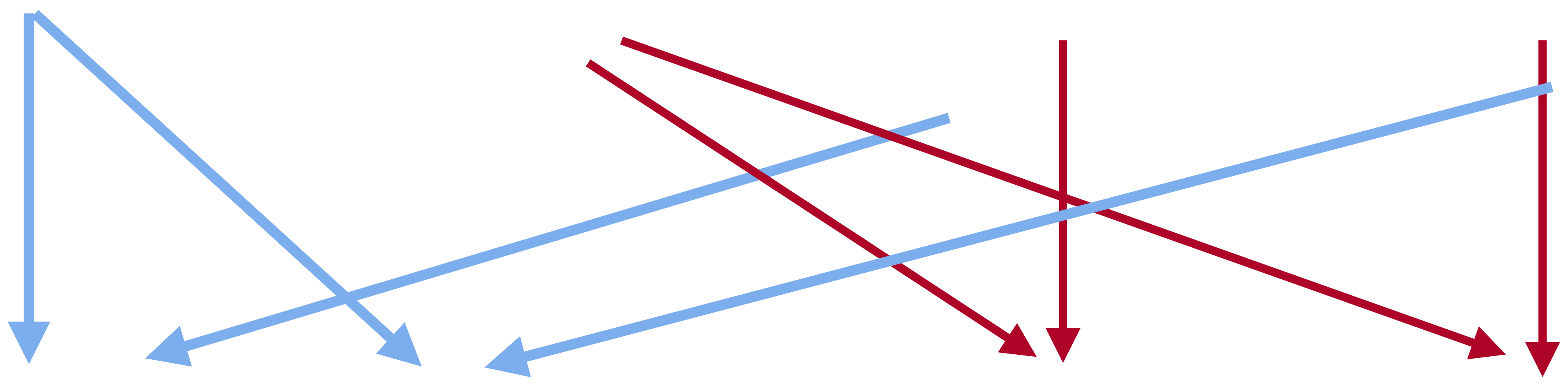
Pick literal p

DP algorithm

$$F = (p \lor q) \land (p \lor \neg q) \land (\neg p \lor r) \land (\neg p \lor \neg r)$$

* No pure literal, no clause with $l \lor \neg l$

Pick literal p



$$(q \lor r) \ \land \ (q \lor \neg r) \ \land \ (\neg q \lor r) \ \land \ (\neg q \lor \neg r)$$

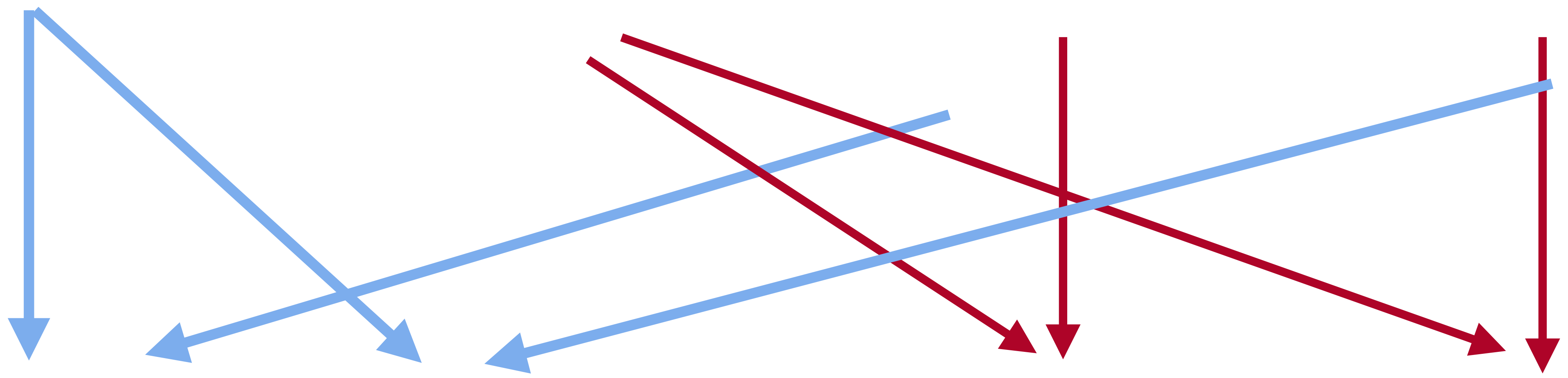
* No pure literal, no clause with $l \lor \neg l$

DP algorithm

$$F = (p \lor q) \land (p \lor \neg q) \land (\neg p \lor r) \land (\neg p \lor \neg r)$$

* No pure literal, no clause with $l \lor \neg l$

  Pick literal p



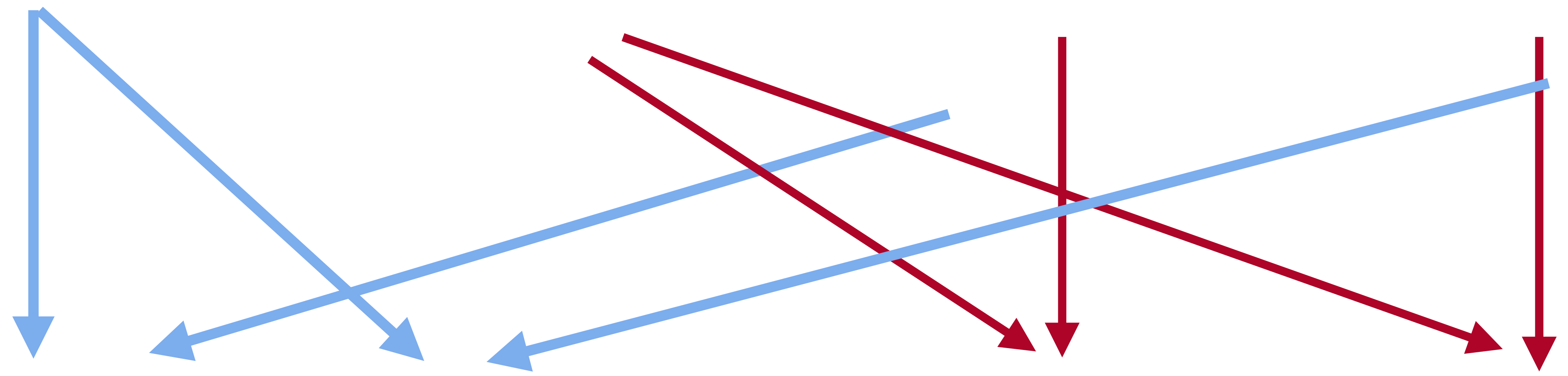
$$(q \lor r) \;\land\; (q \lor \neg r) \;\land\; (\neg q \lor r) \;\land\; (\neg q \lor \neg r)$$

* No pure literal, no clause with $l \lor \neg l$

  Pick literal q

DP algorithm

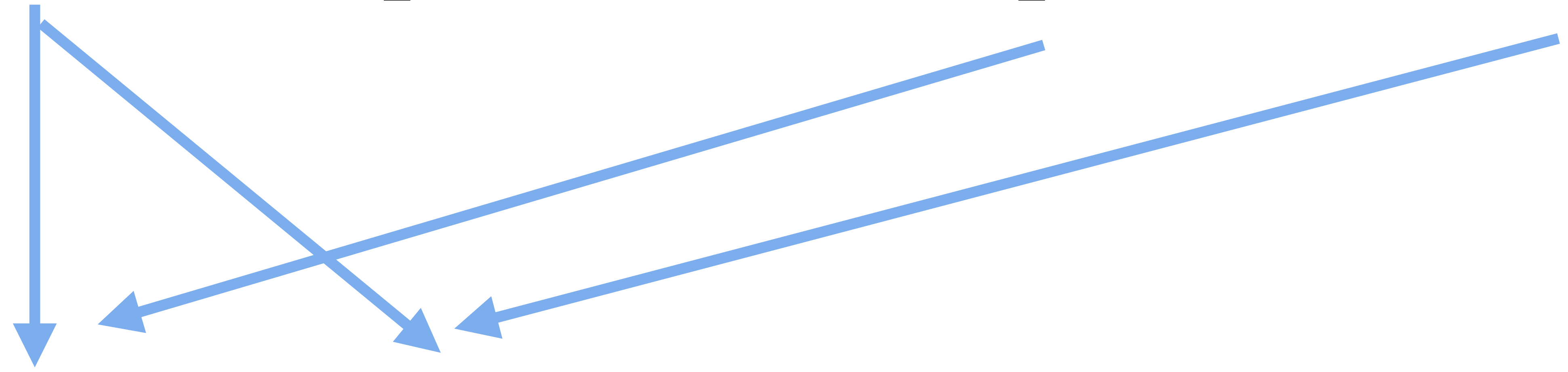
$$F = (p \vee q) \wedge (p \vee \neg q) \wedge (\neg p \vee r) \wedge (\neg p \vee \neg r)$$



* No pure literal, no clause with $l \vee \neg l$

Pick literal p

$$(q \vee r) \;\wedge\; (q \vee \neg r) \;\wedge\; (\neg q \vee r) \;\wedge\; (\neg q \vee \neg r)$$

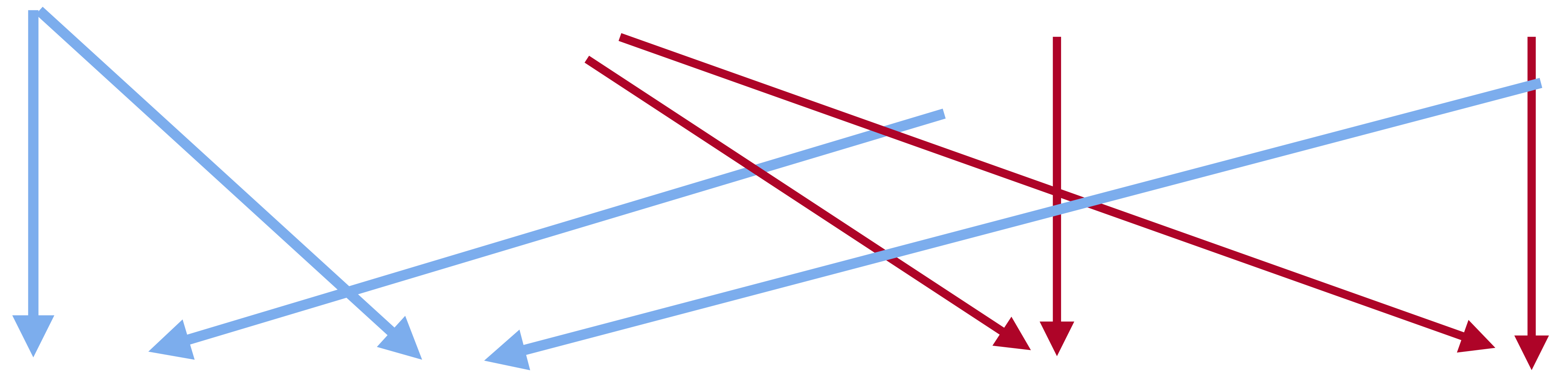


* No pure literal, no clause with $l \vee \neg l$

Pick literal q

DP algorithm

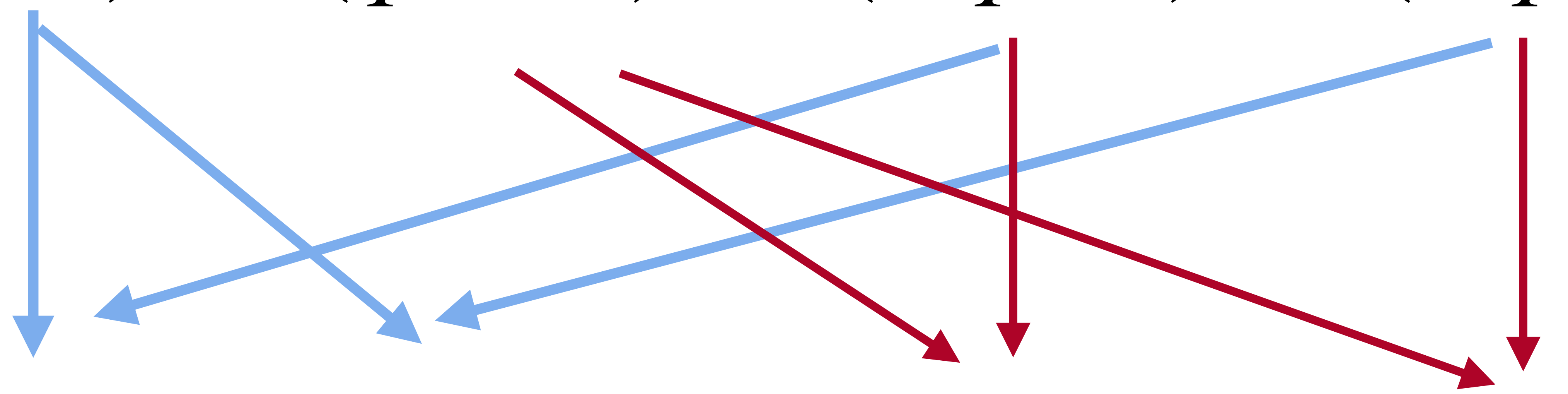
$$F = (p \vee q) \wedge (p \vee \neg q) \wedge (\neg p \vee r) \wedge (\neg p \vee \neg r)$$



* No pure literal, no clause with $l \vee \neg l$

  Pick literal p

$$(q \vee r) \;\wedge\; (q \vee \neg r) \;\wedge\; (\neg q \vee r) \;\wedge\; (\neg q \vee \neg r)$$



* No pure literal, no clause with $l \vee \neg l$

  Pick literal q

DP algorithm

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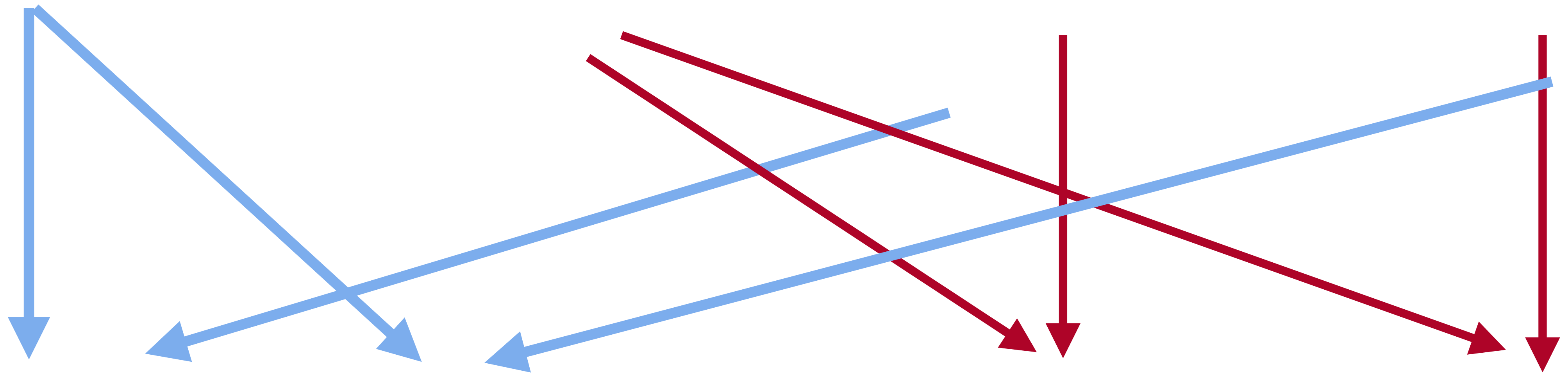
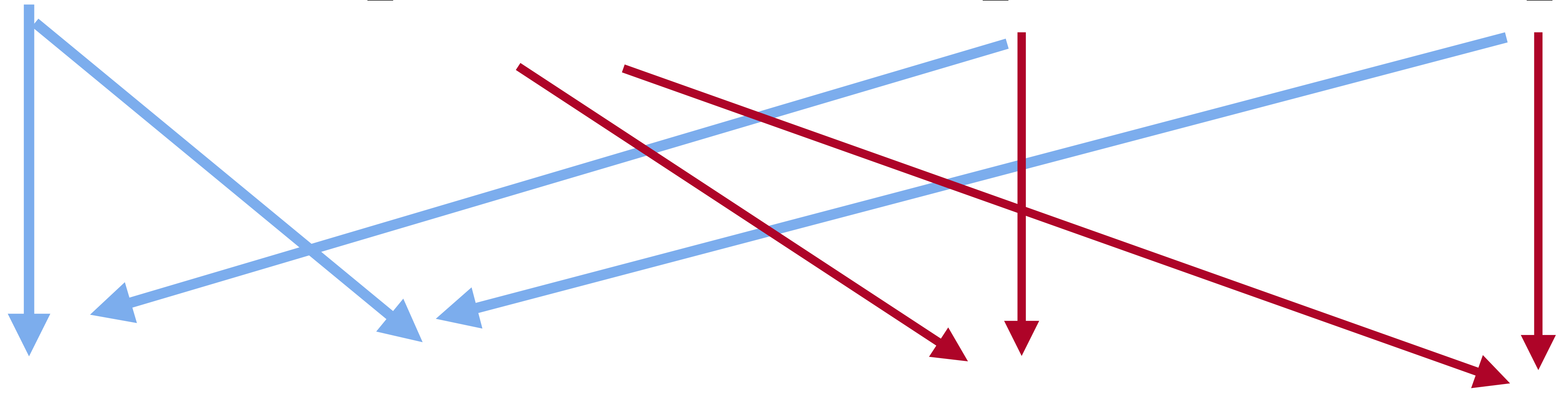


* No pure literal, no clause with $l \vee \neg l$

Pick literal p

$$(q \vee r) \;\wedge\; (q \vee \neg r) \;\wedge\; (\neg q \vee r) \;\wedge\; (\neg q \vee \neg r)$$



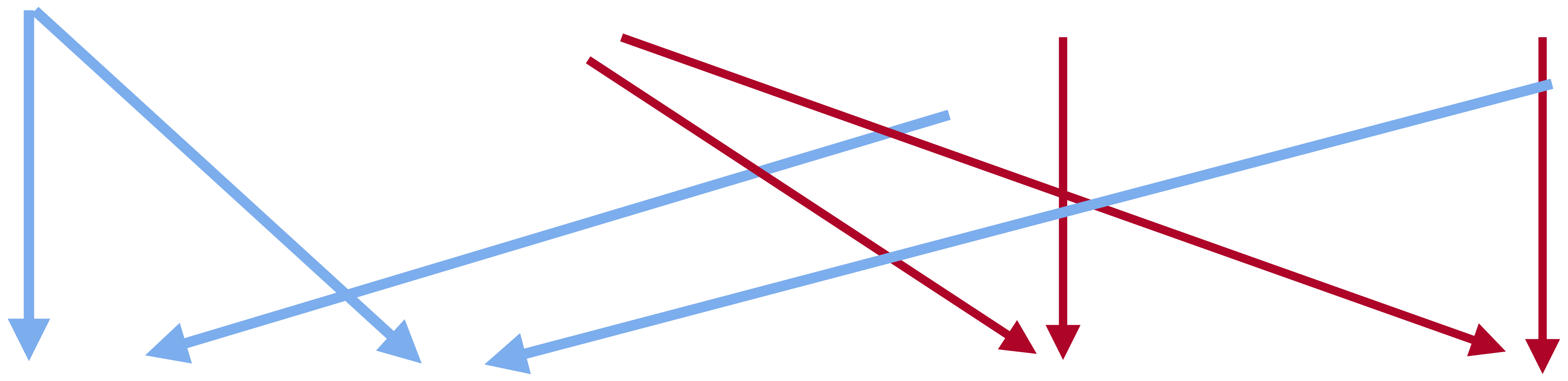
* No pure literal, no clause with $l \vee \neg l$

Pick literal q

$$(r) \;\wedge\; (r \vee \neg r) \;\wedge\; (\neg r \vee r) \;\wedge\; (\neg r)$$

DP algorithm

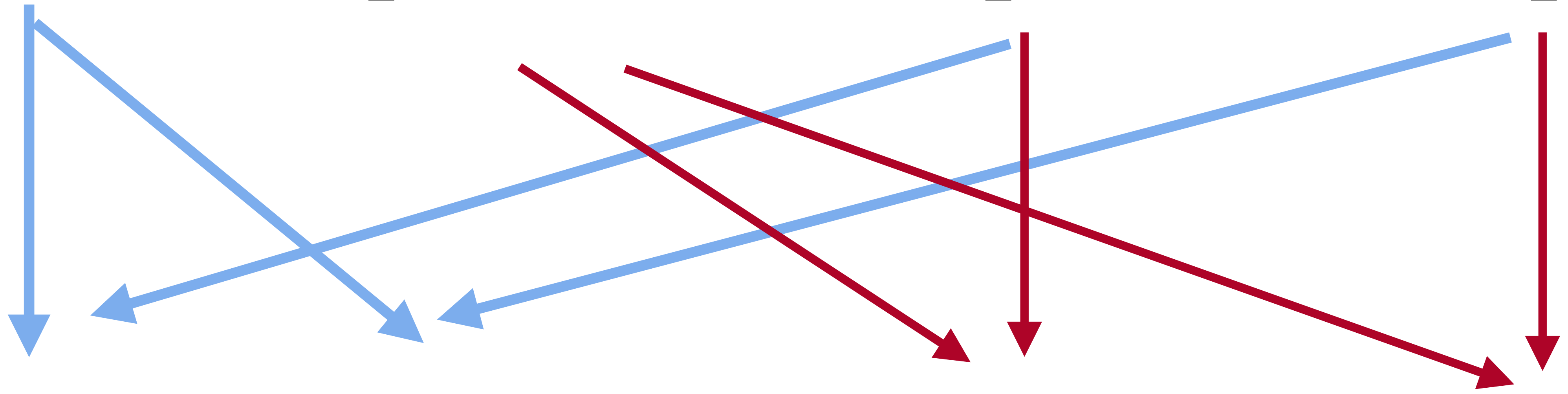
$$F = (p \lor q) \land (p \lor \neg q) \land (\neg p \lor r) \land (\neg p \lor \neg r)$$



* No pure literal, no clause with $l \lor \neg l$

Pick literal p

$$(q \lor r) \ \land \ (q \lor \neg r) \ \land \ (\neg q \lor r) \ \land \ (\neg q \lor \neg r)$$



* No pure literal, no clause with $l \lor \neg l$

Pick literal q

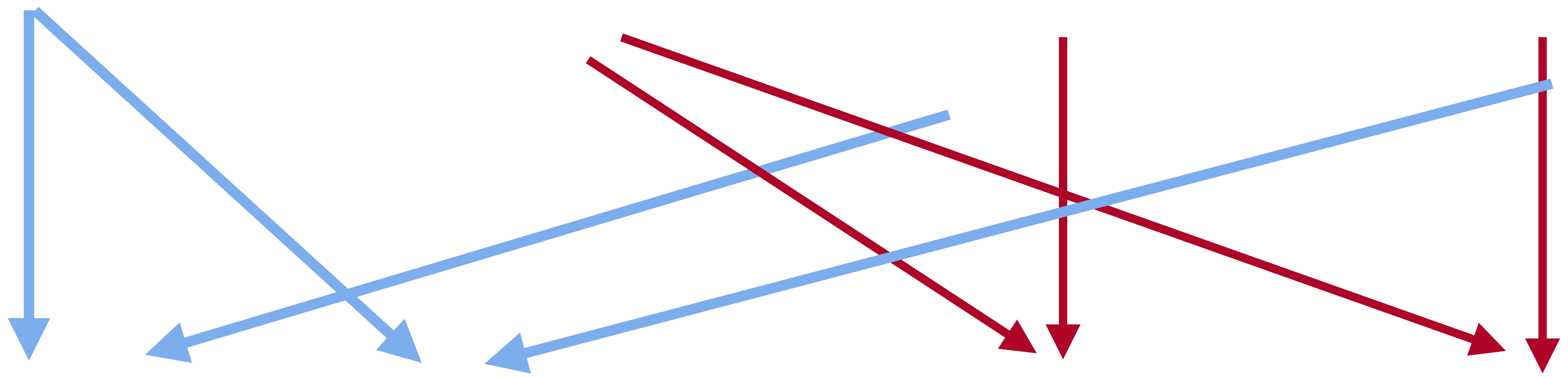
$$(r) \ \land \ (r \lor \neg r) \ \land \ (\neg r \lor r) \ \land \ (\neg r)$$

*remove clauses with $l \lor \neg l$

$$(r) \land (\neg r)$$

DP algorithm

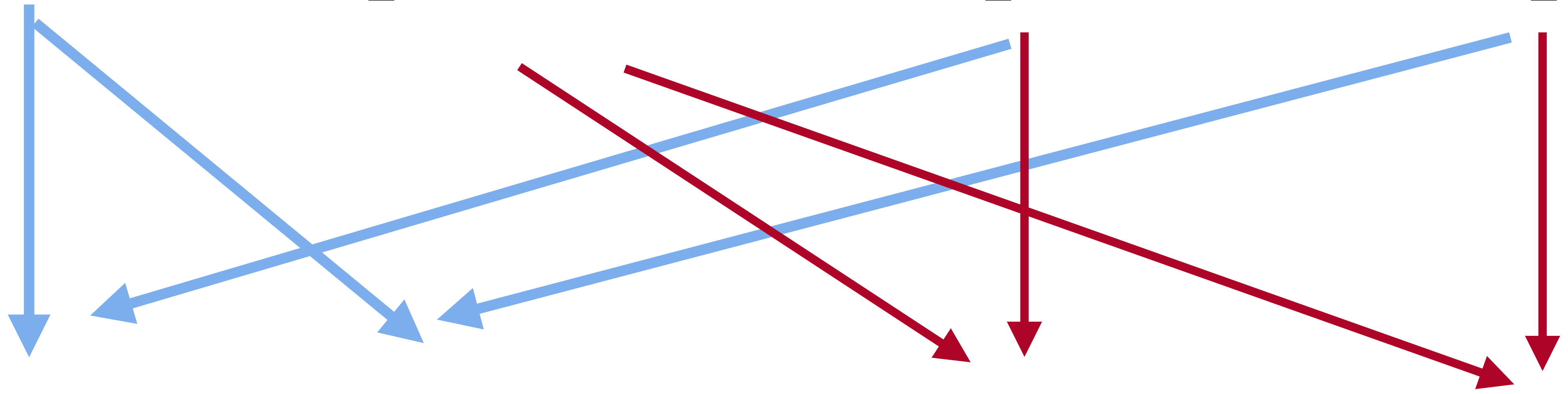
$$F = (p \lor q) \land (p \lor \neg q) \land (\neg p \lor r) \land (\neg p \lor \neg r)$$



* No pure literal, no clause with $l \lor \neg l$

Pick literal p

$$(q \lor r) \;\land\; (q \lor \neg r) \;\land\; (\neg q \lor r) \;\land\; (\neg q \lor \neg r)$$



* No pure literal, no clause with $l \lor \neg l$

Pick literal q

$$(r) \;\land\; (r \lor \neg r) \;\land\; (\neg r \lor r) \;\land\; (\neg r)$$

*remove clauses with $l \lor \neg l$

$$(r) \land (\neg r)$$

Pick literal r

DP algorithm

$$F = (p \lor q) \land (p \lor \neg q) \land (\neg p \lor r) \land (\neg p \lor \neg r)$$

* No pure literal, no clause with $l \lor \neg l$

Pick literal p

$$(q \lor r) \;\land\; (q \lor \neg r) \;\land\; (\neg q \lor r) \;\land\; (\neg q \lor \neg r)$$

* No pure literal, no clause with $l \lor \neg l$

Pick literal q

$$(r) \;\land\; (r \lor \neg r) \;\land\; (\neg r \lor r) \;\land\; (\neg r)$$

*remove clauses with $l \lor \neg l$

$$(r) \land (\neg r)$$

Pick literal r

$$\square$$

DP algorithm

$$F = (p \vee q) \wedge (p \vee \neg q) \wedge (\neg p \vee r) \wedge (\neg p \vee \neg r)$$

* No pure literal, no clause with $l \vee \neg l$

Pick literal p

$$(q \vee r) \;\wedge\; (q \vee \neg r) \;\wedge\; (\neg q \vee r) \;\wedge\; (\neg q \vee \neg r)$$

* No pure literal, no clause with $l \vee \neg l$

Pick literal q

$$(r) \;\wedge\; (r \vee \neg r) \;\wedge\; (\neg r \vee r) \;\wedge\; (\neg r)$$

*remove clauses with $l \vee \neg l$

$$(r) \wedge (\neg r)$$

Pick literal r

$$\square$$

F has empty clause — UNSAT

DP algorithm

$$F = (p \vee q \vee r) \wedge (q \vee \neg r \vee \neg s) \wedge (\neg q \vee s) \wedge (\neg p \vee \neg s)$$

DP algorithm

$$F = (p \lor q \lor r) \land (q \lor \neg r \lor \neg s) \land (\neg q \lor s) \land (\neg p \lor \neg s)$$

$$(q \lor r \lor \neg s) \land (q \lor \neg r \lor \neg s) \land (\neg q \lor s)$$

$$(r \lor \neg s \lor s) \land (\neg r \lor \neg s \lor s)$$

$$\{\}$$

Empty Formula, return SAT

DP algorithm

$$F = (p \lor q \lor r) \land (q \lor \neg r \lor \neg s) \land (\neg q \lor s) \land (\neg p \lor \neg s)$$

* No pure literal, no clause with $l \lor \neg l$

$$(q \lor r \lor \neg s) \land (q \lor \neg r \lor \neg s) \land (\neg q \lor s)$$

$$(r \lor \neg s \lor s) \land (\neg r \lor \neg s \lor s)$$

$$\{\}$$

Empty Formula, return SAT

DP algorithm

$$F = (p \vee q \vee r) \wedge (q \vee \neg r \vee \neg s) \wedge (\neg q \vee s) \wedge (\neg p \vee \neg s)$$

* No pure literal, no clause with $l \vee \neg l$

Pick literal p

$$(q \vee r \vee \neg s) \wedge (q \vee \neg r \vee \neg s) \wedge (\neg q \vee s)$$

$$(r \vee \neg s \vee s) \wedge (\neg r \vee \neg s \vee s)$$

$$\{\}$$

Empty Formula, return SAT

DP algorithm

$$F = (p \vee q \vee r) \wedge (q \vee \neg r \vee \neg s) \wedge (\neg q \vee s) \wedge (\neg p \vee \neg s)$$

* No pure literal, no clause with $l \vee \neg l$
  Pick literal p

$$(q \vee r \vee \neg s) \wedge (q \vee \neg r \vee \neg s) \wedge (\neg q \vee s)$$

* No pure literal, no clause with $l \vee \neg l$

$$(r \vee \neg s \vee s) \wedge (\neg r \vee \neg s \vee s)$$

$$\{\}$$

Empty Formula, return SAT

DP algorithm

$$F = (p \vee q \vee r) \wedge (q \vee \neg r \vee \neg s) \wedge (\neg q \vee s) \wedge (\neg p \vee \neg s)$$

* No pure literal, no clause with $l \vee \neg l$

Pick literal p

$$(q \vee r \vee \neg s) \wedge (q \vee \neg r \vee \neg s) \wedge (\neg q \vee s)$$

* No pure literal, no clause with $l \vee \neg l$

Pick literal q

$$(r \vee \neg s \vee s) \wedge (\neg r \vee \neg s \vee s)$$

$$\{\}$$

Empty Formula, return SAT

DP algorithm

$$F = (p \vee q \vee r) \wedge (q \vee \neg r \vee \neg s) \wedge (\neg q \vee s) \wedge (\neg p \vee \neg s)$$

* No pure literal, no clause with $l \vee \neg l$

  Pick literal p

$$(q \vee r \vee \neg s) \wedge (q \vee \neg r \vee \neg s) \wedge (\neg q \vee s)$$

* No pure literal, no clause with $l \vee \neg l$

  Pick literal q

$$(r \vee \neg s \vee s) \wedge (\neg r \vee \neg s \vee s)$$

* remove clauses with $l \vee \neg l$

$$\{\}$$

Empty Formula, return SAT

$$F = (p \vee q) \wedge (p \vee \neg q) \wedge (\neg p \vee r) \wedge (\neg p \vee \neg r)$$

$$F = (p \vee q) \wedge (p \vee \neg q) \wedge (\neg p \vee r) \wedge (\neg p \vee \neg r) \;\; \wedge \neg p$$

$$F = (p \lor q) \land (p \lor \neg q) \land (\neg p \lor r) \land (\neg p \lor \neg r) \land \neg p$$

Unit clause

$$F = (p \lor q) \land (p \lor \neg q) \land \underline{(\neg p \lor r) \land (\neg p \lor \neg r) \land \neg p}$$

Unit clause

$$F = (p \lor q) \land (p \lor \neg q) \land (\neg p \lor r) \land (\neg p \lor \neg r) \land \neg p$$

Unit clause

p has to take value 0, $(\neg p \lor r) \land (\neg p \lor \neg r)$ are True

$$F = (p \lor q) \land (p \lor \neg q) \land (\neg p \lor r) \land (\neg p \lor \neg r) \land \neg p$$ Unit clause

p has to take value 0, $(\neg p \lor r) \land (\neg p \lor \neg r)$ are True

Can we remove all clauses that have $\neg p$?

$$F = (p \lor q) \land (p \lor \neg q) \land (\neg p \lor r) \land (\neg p \lor \neg r) \land \neg p$$ Unit clause

p has to take value 0, $(\neg p \lor r) \land (\neg p \lor \neg r)$ are True

Can we remove all clauses that have $\neg p$?

$$F = (p \vee q) \wedge (p \vee \neg q) \wedge (\neg p \vee r) \wedge (\neg p \vee \neg r) \wedge \neg p$$

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Can we remove all clauses that have $\neg p$?

$$F = (p \vee q) \wedge (p \vee \neg q) \wedge (\neg p \vee r) \wedge (\neg p \vee \neg r) \wedge \neg p$$

Unit clause

p has to take value 0, $(\neg p \vee r) \wedge (\neg p \vee \neg r)$ are True

Can we remove all clauses that have $\neg p$?

$$(\neg p) \wedge (p \vee q) \equiv_{SAT} q$$

$$F = (p \lor q) \land (p \lor \neg q) \land (\neg p \lor r) \land (\neg p \lor \neg r) \land \neg p$$

Unit clause

p has to take value 0, $(\neg p \lor r) \land (\neg p \lor \neg r)$ are True

Can we remove all clauses that have $\neg p$?

$$(\neg p) \land (p \lor q) \equiv_{SAT} q$$

$$(\neg p) \land (p \lor \neg q) \equiv_{SAT} \neg q$$

Unit Propagation

While F contains a unit clause (*l*) do:

For every clause C in F that has *l* do:

$$F_{CNF} \leftarrow remove\_from\_formula(C, F_{CNF})$$

For every clause C in F that has $\neg l$ do:

$$F_{CNF} \leftarrow remove\_from\_formula(C, F_{CNF})$$

$$F_{CNF} \leftarrow add\_to\_formula(C \setminus \neg l, F_{CNF})$$

DP algorithm for SAT Solving (Martin Davis - Hilary Putnam 1960)

1. Start with F_{CNF}
2. For every clause C in F_{CNF} that either contains both l and $\neg l$ or has pure literal do:
   1. $F_{CNF} \leftarrow remove\_from\_formula(C, F_{CNF})$
3. $F_{CNF} \leftarrow$ UnitPropagation(F_{CNF})
4. If F_{CNF} is empty
   1. Return SAT
5. If F_{CNF} has empty clause then
   1. Return UNSAT
6. Pick a literal l that occurs with both polarities in F_{CNF}.
   1. $F_{CNF} \leftarrow Resolution(C, l, F_{CNF})$
7. For every clause C that contains l or $\neg l$ do :
   1. $F_{CNF} \leftarrow remove\_from\_formula(C, F_{CNF})$

DPLL algorithm (Davis -Putnam-Logemann-Loveland 1960)

Complete and Sound algorithm & takes linear space in worst case.

Still the basis of SAT solver

zChaff Solver — efficient implementation of DPLL.

Won test of time award at CAV 2001.

Notations

Partial Model: subset of elements of Vars(F) maps to $\{0,1\}$

Under partial model m,

A literal l is True if $m(l)$= 1

A literal l is False if $m(l)$ = 0

Otherwise:

l is unassigned.

Example: $F = (x_1 \vee x_2 \vee \neg x_3); m = \{x_1 \mapsto 1, x_3 \mapsto 1\}$

x_1 is True, x_2 is unassigned, x_3 is False.

Notations

Partial Model: subset of elements of Vars(F) maps to {0,1}

Under partial model m,

Clause C is True if there is a $l \in C$, such that l is True.

Clause C is False if for each literal $l \in C$, l is False

Otherwise:

C is unassigned.

Example: $m = \{x_1 \mapsto 1, x_3 \mapsto 1\}$

Notations

Partial Model: subset of elements of Vars(F) maps to $\{0,1\}$

Under partial model m,

Clause C is True if there is a $l \in C$, such that l is True.

Clause C is False if for each literal $l \in C$, l is False

Otherwise:

C is unassigned.

$C = (x_1 \vee x_2 \vee x_3)$ — True

Example: $m = \{x_1 \mapsto 1, x_3 \mapsto 1\}$

Notations

Partial Model: subset of elements of Vars(F) maps to $\{0,1\}$

Under partial model m,

Clause C is True if there is a $l \in C$, such that l is True.

Clause C is False if for each literal $l \in C$, l is False

Otherwise:

C is unassigned.

Example: $m = \{x_1 \mapsto 1, x_3 \mapsto 1\}$

$C = (x_1 \vee x_2 \vee x_3)$ — True

$C = (\neg x_1 \vee x_2 \vee \neg x_3)$ — Unassigned

Notations

Partial Model: subset of elements of Vars(F) maps to $\{0,1\}$

Under partial model m,

Clause C is True if there is a $l \in C$, such that l is True.

Clause C is False if for each literal $l \in C$, l is False

Otherwise:

C is unassigned.

Example: $m = \{x_1 \mapsto 1, x_3 \mapsto 1\}$

$C = (x_1 \vee x_2 \vee x_3)$ — True

$C = (\neg x_1 \vee x_2 \vee \neg x_3)$ — Unassigned

$C = (\neg x_1 \vee \neg x_3)$ — False

Notations

Partial Model: subset of elements of Vars(F) maps to $\{0,1\}$

Under partial model m,

F_{CNF} is True if for each $C \in F_{CNF}$, C is True.

F_{CNF} is False if there is a $C \in F_{CNF}$ such that C is False

Otherwise:

F_{CNF} is unassigned.

Notations

Partial Model: subset of elements of Vars(F) maps to $\{0,1\}$

Under partial model m,

F_{CNF} is True if for each $C \in F_{CNF}$, C is True.

F_{CNF} is False if there is a $C \in F_{CNF}$ such that C is False

Otherwise:

F_{CNF} is unassigned.

Unit Clause (updated): C is a unit clause under partial model m if there is exactly one literal l in C which is unassigned, and rest all literals of C are False.

Notations

Partial Model: subset of elements of Vars(F) maps to $\{0,1\}$

Under partial model m,

F_{CNF} is True if for each $C \in F_{CNF}$, C is True.

F_{CNF} is False if there is a $C \in F_{CNF}$ such that C is False

Otherwise:

F_{CNF} is unassigned.

Unit Clause (updated): C is a unit clause under partial model m if there is exactly one literal l in C which is unassigned, and rest all literals of C are False.

Example: $C = (x_1 \vee \neg x_3 \vee \neg x_2); m = \{x_1 \mapsto 0, x_2 \mapsto 1\}$

Notations

Partial Model: subset of elements of Vars(F) maps to $\{0,1\}$

Under partial model m,

F_{CNF} is True if for each $C \in F_{CNF}$, C is True.

F_{CNF} is False if there is a $C \in F_{CNF}$ such that C is False

Otherwise:

F_{CNF} is unassigned.

Unit Clause (updated): C is a unit clause under partial model m if there is exactly one literal l in C which is unassigned, and rest all literals of C are False.

Example: $C = (x_1 \vee \neg x_3 \vee \neg x_2); m = \{x_1 \mapsto 0, x_2 \mapsto 1\}$ C is unit clause under m.

DPLL algorithm (Davis -Putnam-Logemann-Loveland 1960)

1. Maintains a partial model, initially $\varnothing$
2. Assign unassigned variables either 0 or 1
    1. (Randomly one after the other)
3. Sometime forced to make a decision due to unit clause

DPLL algorithm (Davis -Putnam-Logemann-Loveland 1960)

$F = (x_1 \vee \neg x_2) \wedge (\neg x_1 \vee x_2 \vee \neg x_3)$

Initially m is $\emptyset$

Pick a variable, say x_3, and assign it a Boolean value, say 1. Partial model $m = \{x_3 \mapsto 1\}$

$C_1 : (x_1 \vee \neg x_2)$ unassigned.

$C_2 : (\neg x_1 \vee x_2 \vee \neg x_3)$ unassigned.

Pick another variable, say x_1, and assign it a Boolean value, say 0.

Partial model $m = \{x_1 \mapsto 0, x_3 \mapsto 1\}$

$C_1 : (x_1 \vee \neg x_2)$ Unit clause, forced decision $(x_2 \mapsto 0)$

$C_2 : (\neg x_1 \vee x_2 \vee \neg x_3)$ True.

$m = \{x_1 \mapsto 0, x_2 \mapsto 0, x_3 \mapsto 1\}$ and $m \models F$

DPLL algorithm (Davis -Putnam-Logemann-Loveland 1960)

1. Maintains a partial model, initially $\emptyset$
2. Assign unassigned variables either 0 or 1
    1. (Randomly one after the other)
3. Sometime forced to make a decision due to unit clause

What to do if F is False under partial model m?

Backtracking

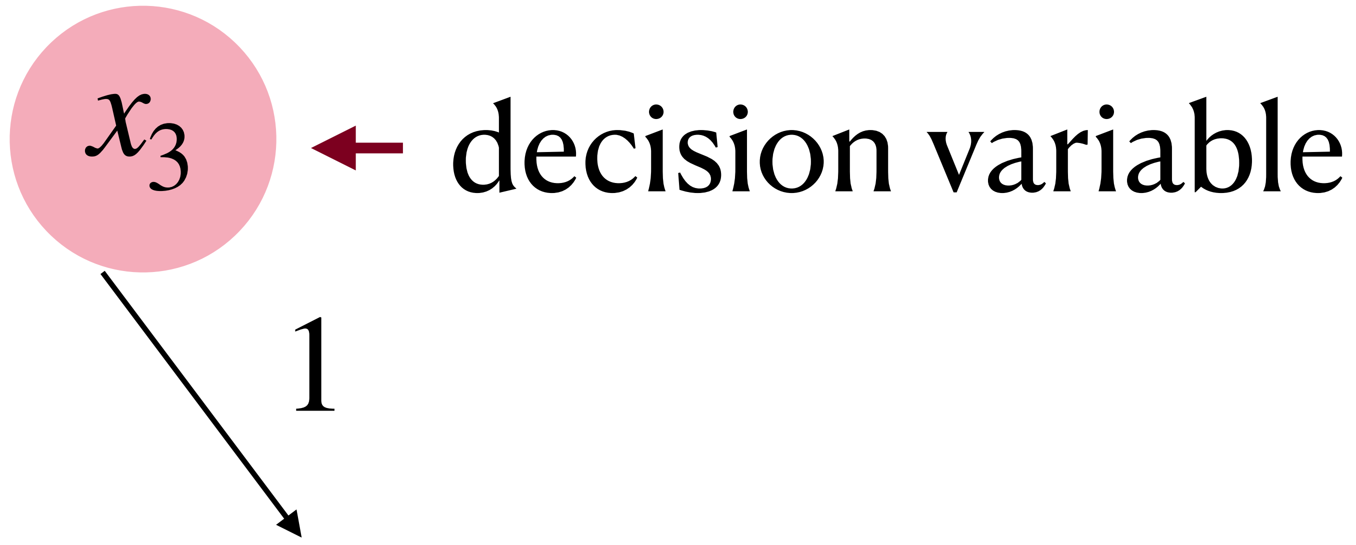
$$F = (\neg x_1 \lor \neg x_2 \lor \neg x_3) \land (\neg x_3 \lor x_2) \land (\neg x_3 \lor x_1)$$

Backtracking

$$F = (\neg x_1 \vee \neg x_2 \vee \neg x_3) \wedge (\neg x_3 \vee x_2) \wedge (\neg x_3 \vee x_1)$$

Pick a variable, say x_3, and assign it a Boolean value, say 1.

Partial model $m = \{x_3 \mapsto 1\}$



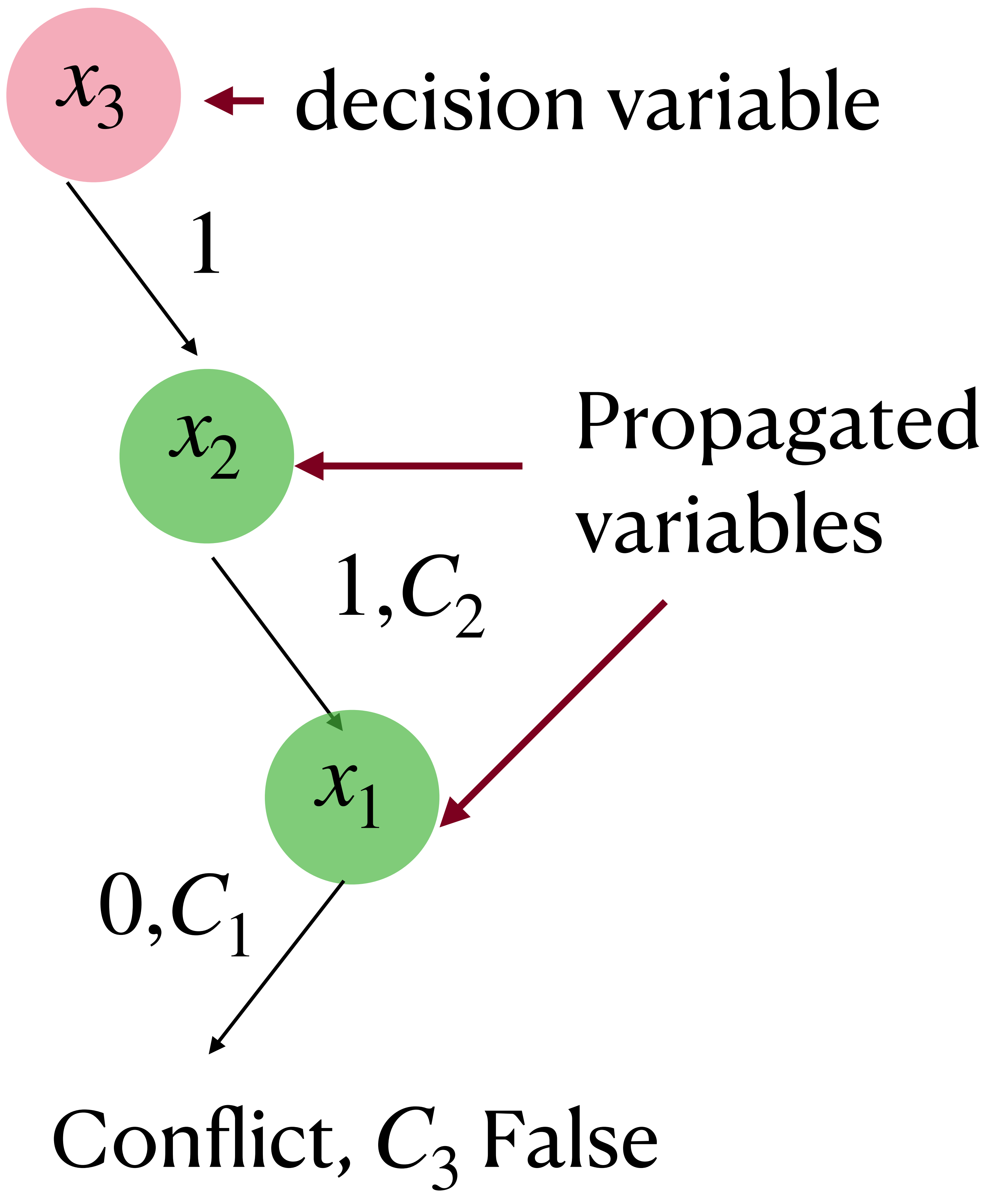
Backtracking

$F = (\neg x_1 \vee \neg x_2 \vee \neg x_3) \wedge (\neg x_3 \vee x_2) \wedge (\neg x_3 \vee x_1)$

Pick a variable, say x_3, and assign it a Boolean value, say 1.

Partial model $m = \{x_3 \mapsto 1\}$

$(\neg x_3 \vee x_2) \wedge (\neg x_3 \vee x_1)$ — unit clauses



Backtracking

$F = (\neg x_1 \lor \neg x_2 \lor \neg x_3) \land (\neg x_3 \lor x_2) \land (\neg x_3 \lor x_1)$

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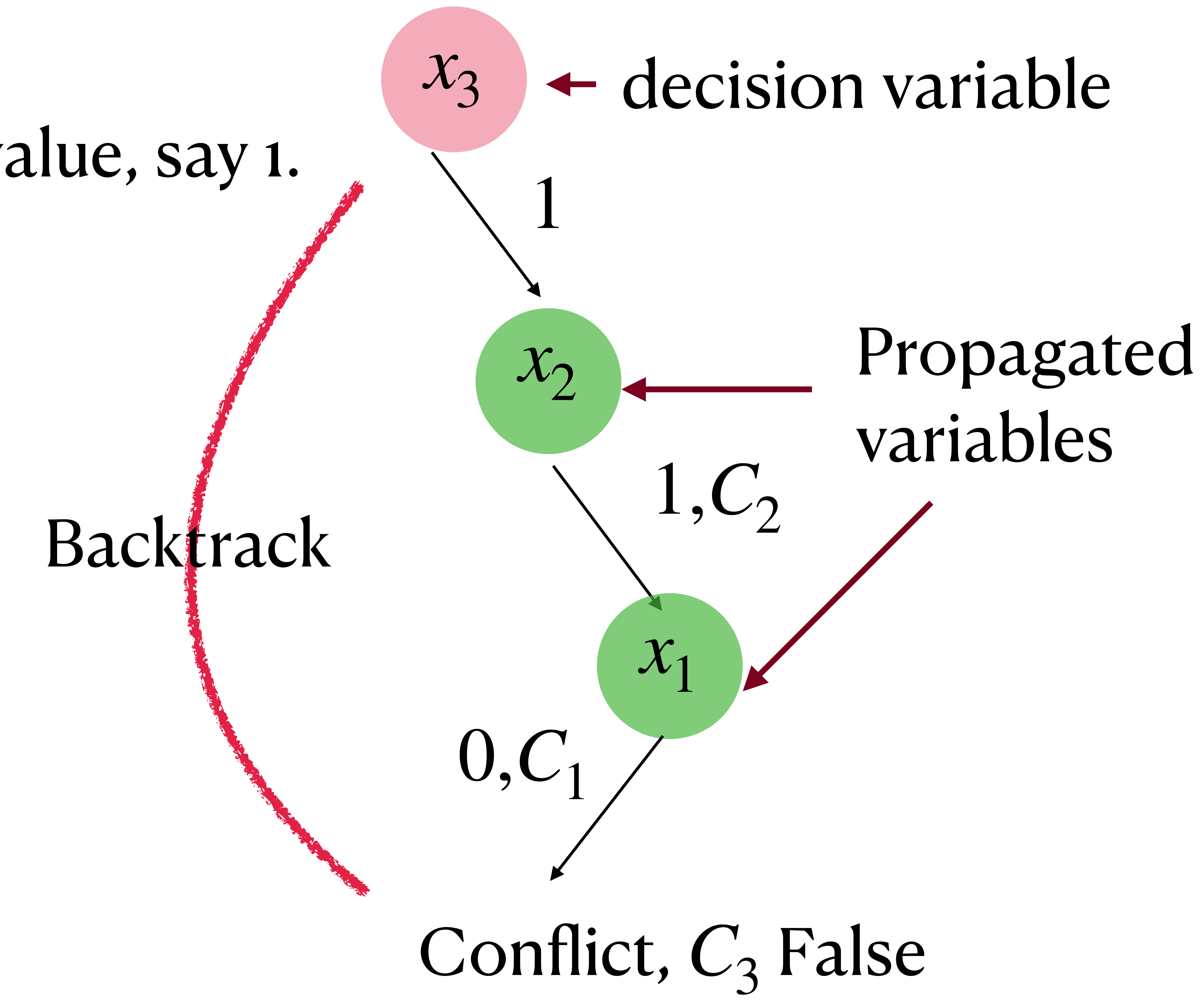

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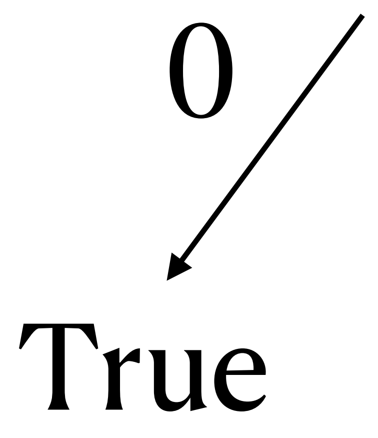

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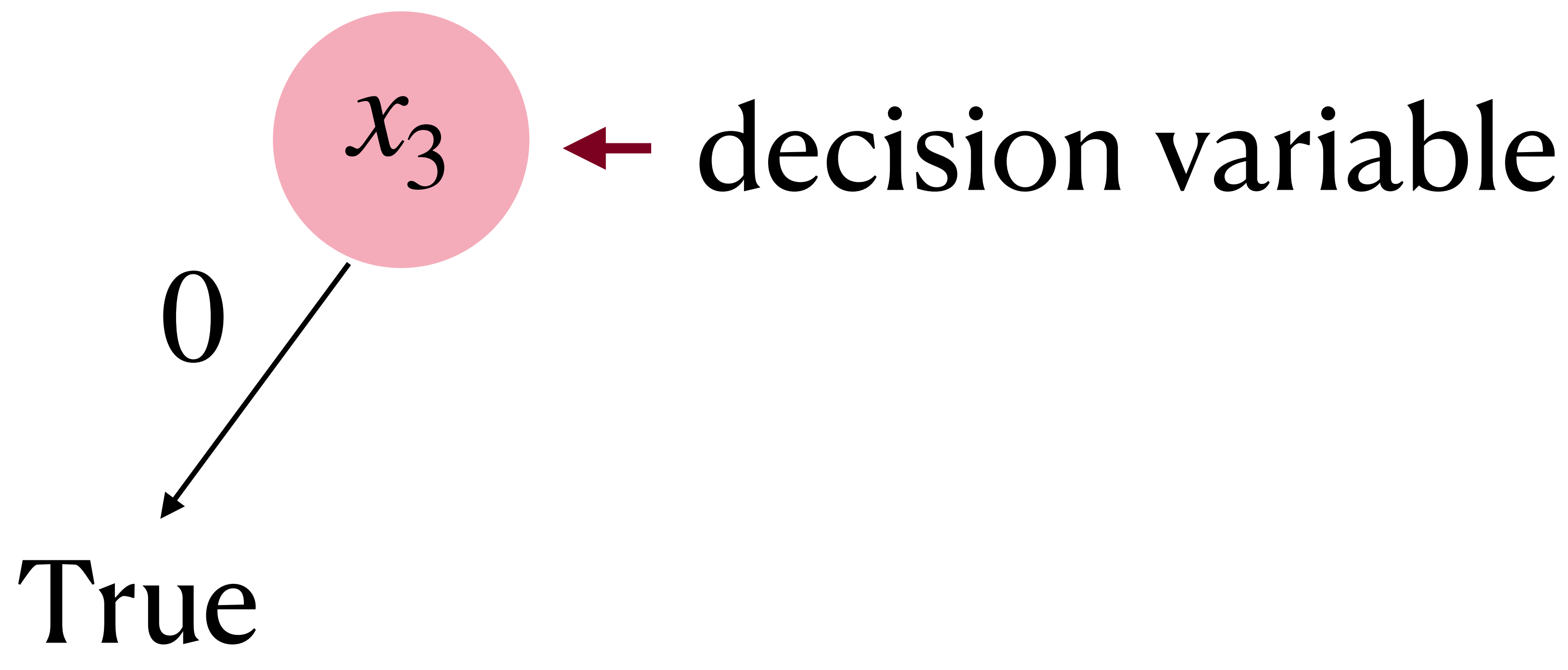

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Backtrack to last decision, and change the polarity.

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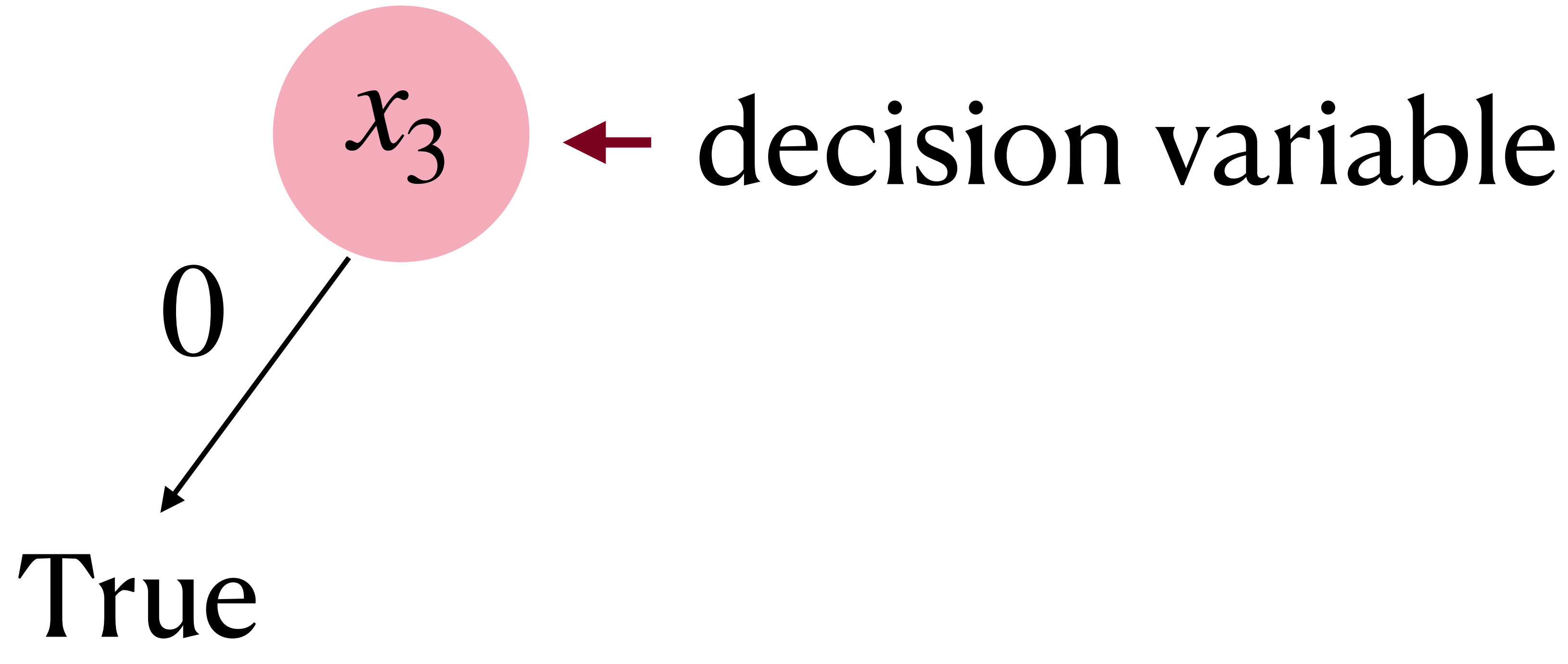

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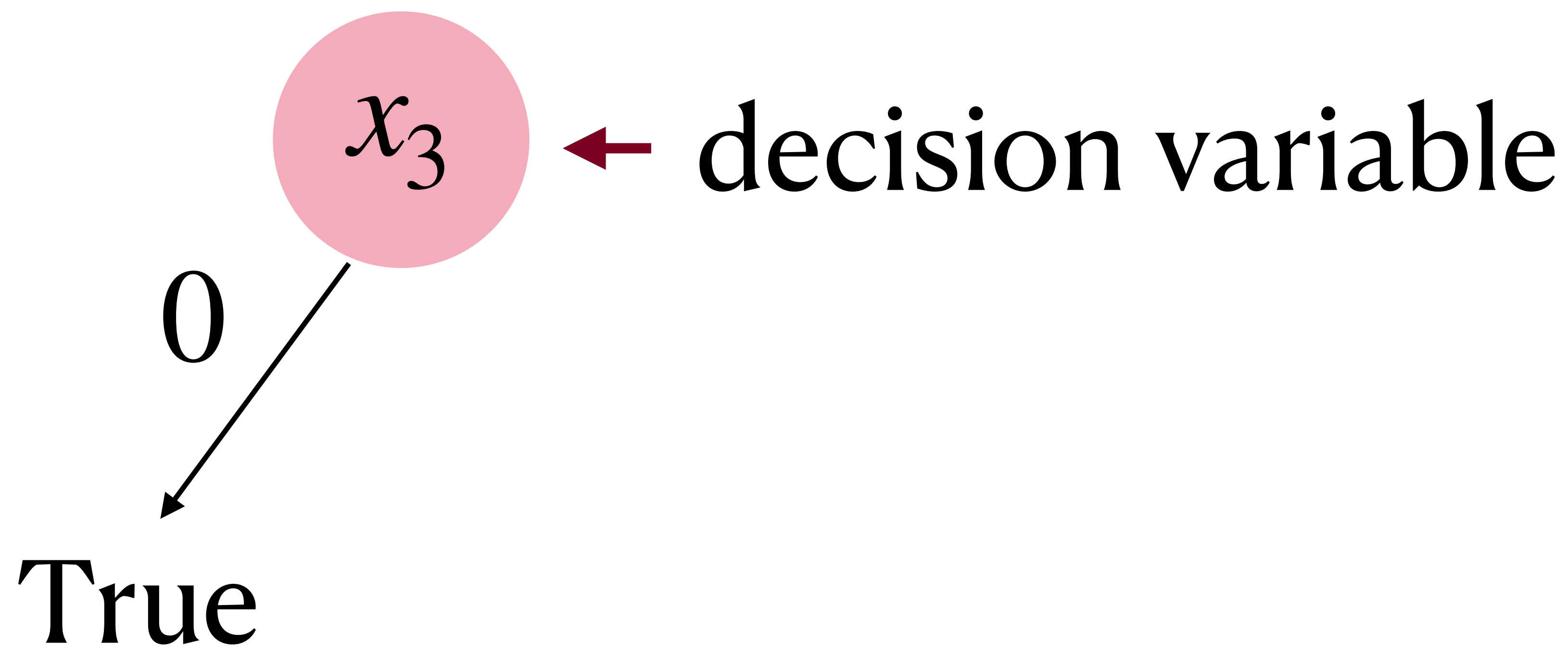
All clauses are True, hence F is True



DPLL algorithm (Davis -Putnam-Logemann-Loveland 1960)

$DPLL(F, m = \emptyset)\{$

1. If F is True under m then Return SAT
2. If F is False under m then Return UNSAT
3. If there is a unit literal l under m then Return $DPLL(F, m[l \mapsto 1])$
4. If there is a unit literal $\neg l$ under m then Return $DPLL(F, m[l \mapsto 0])$

   Choose an unassigned variable p, and random bit $b \in \{0,1\}$

5. If $DPLL(F, m[p \mapsto b]) == $ SAT then Return SAT

   Else Return $DPLL(F, m[p \mapsto 1 - b])$

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$\}$

Backtracking at conflict

Unit Propagation

DPLL algorithm (Davis -Putnam-Logemann-Loveland 1960)

1. Maintains a partial model, initially $\emptyset$
2. Assign unassigned variables either 0 or 1
    1. (Randomly one after the other)
3. Sometime forced to make a decision due to unit clause

DPLL run consists of

- Decision
- Unit propagation
- Backtracking

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decision variable

propagated variables

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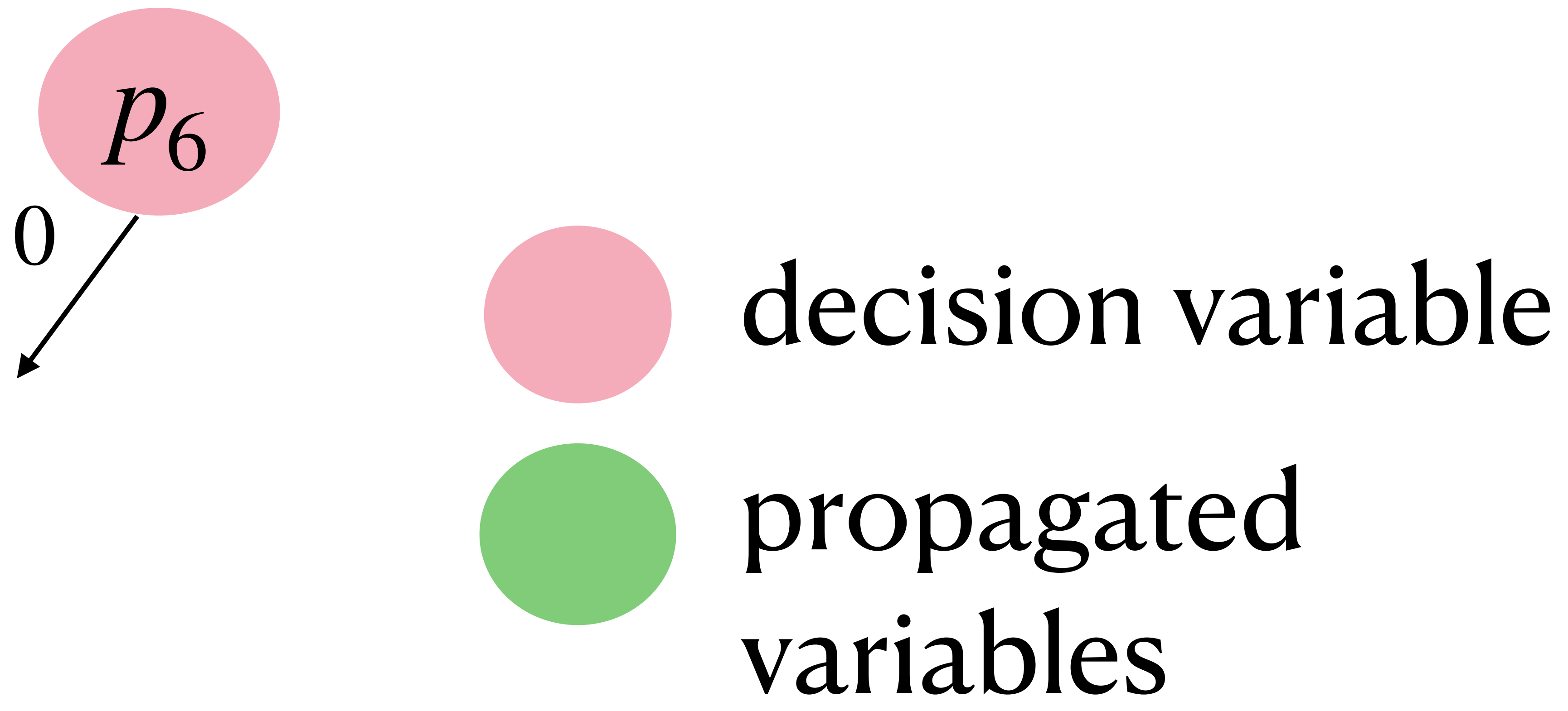

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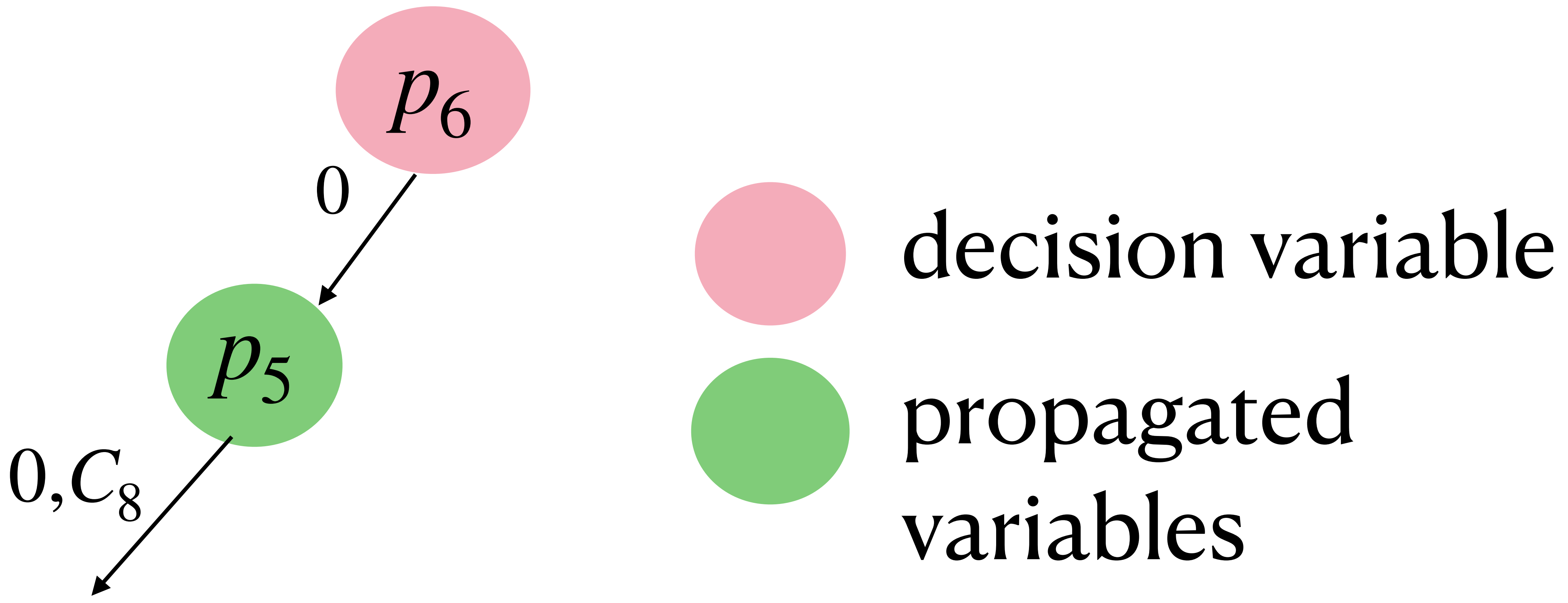

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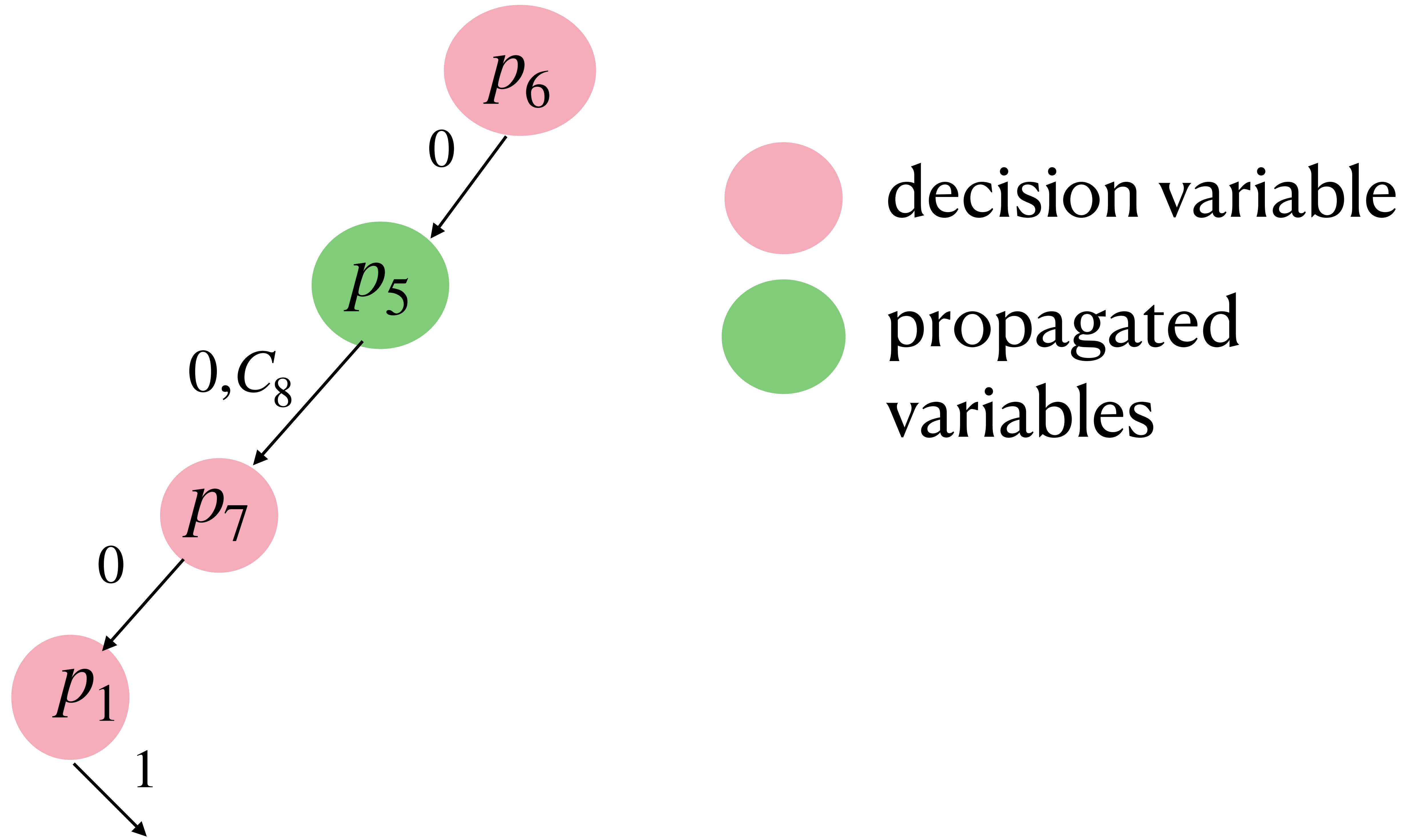

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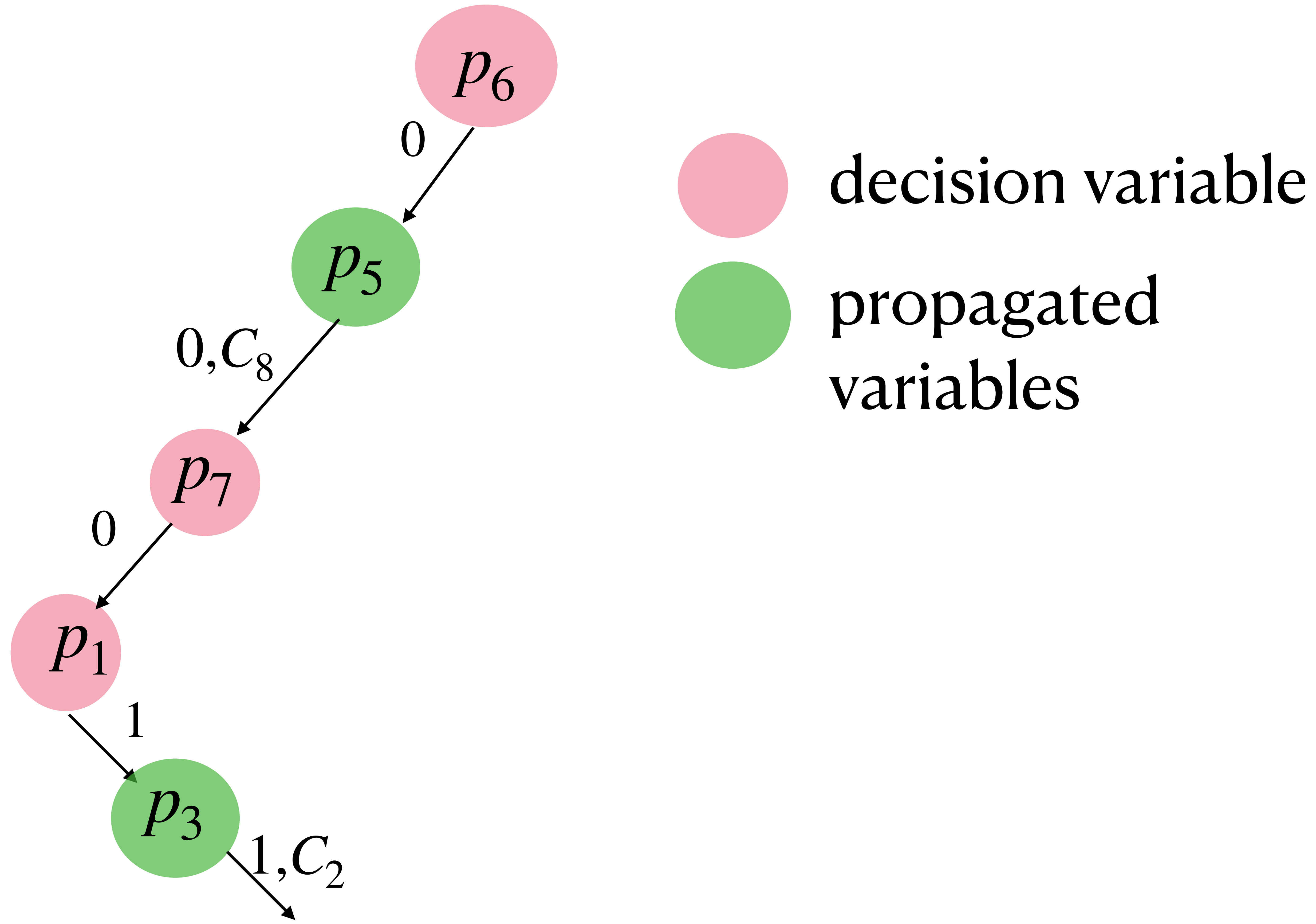

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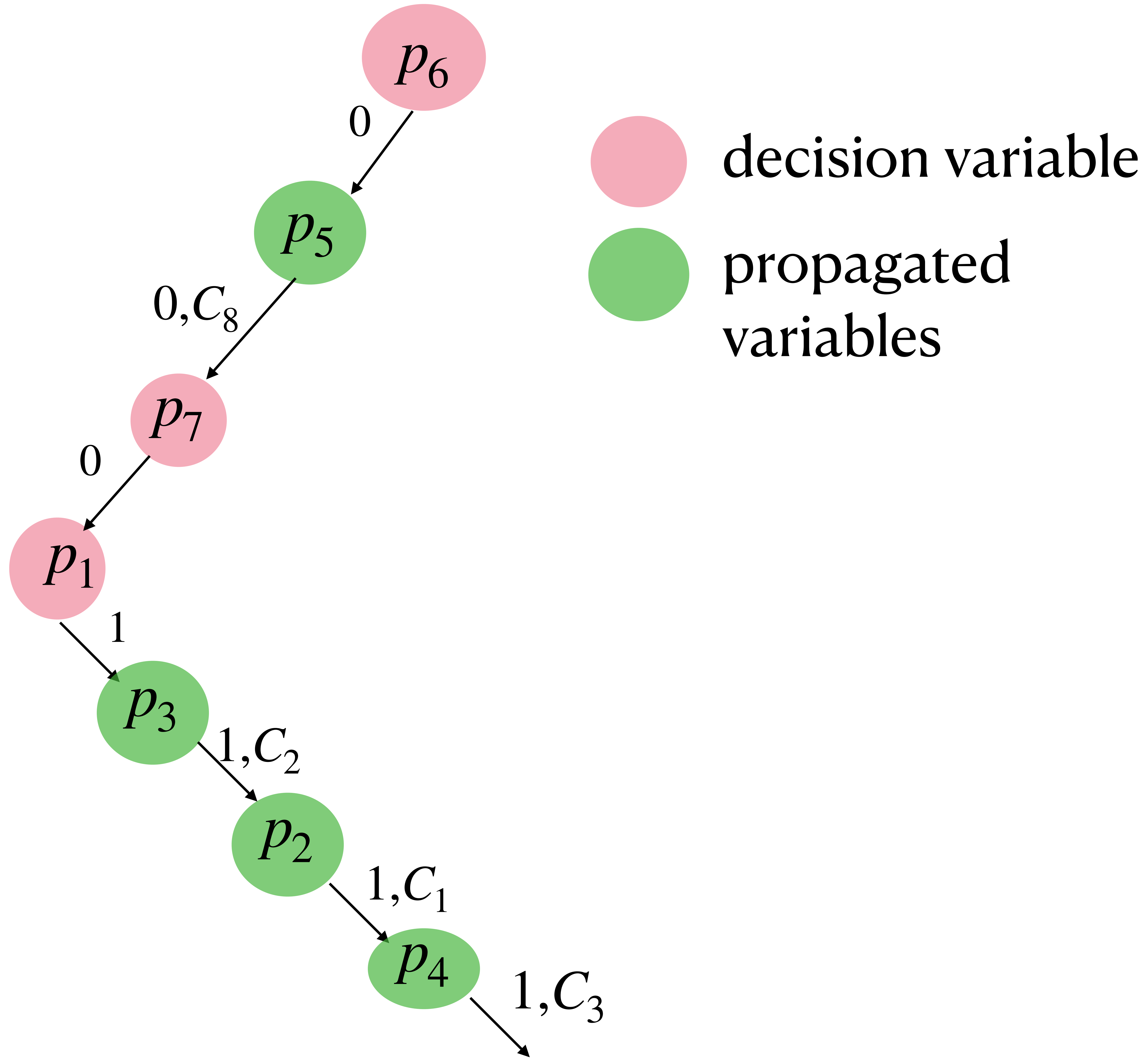

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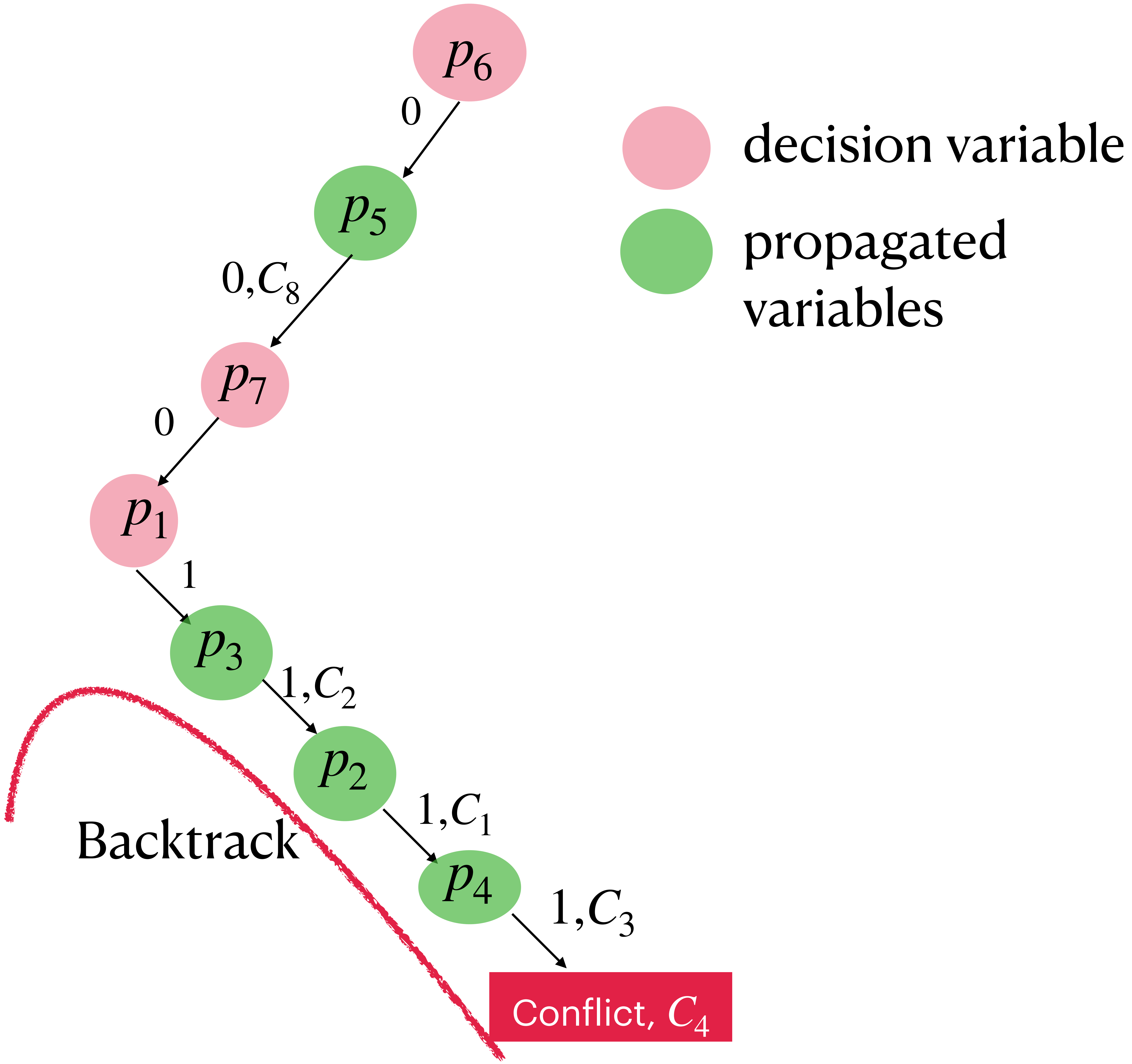

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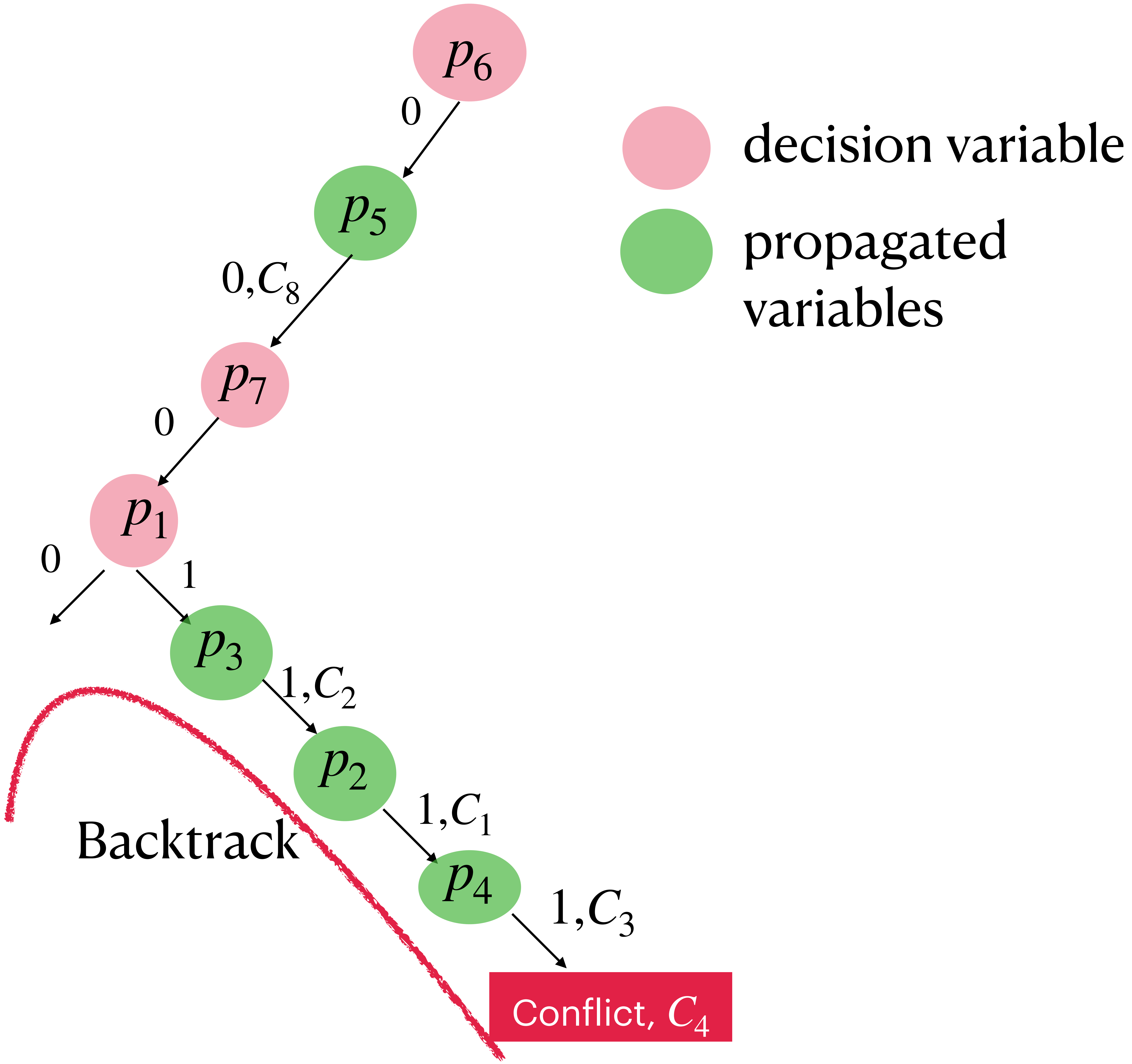

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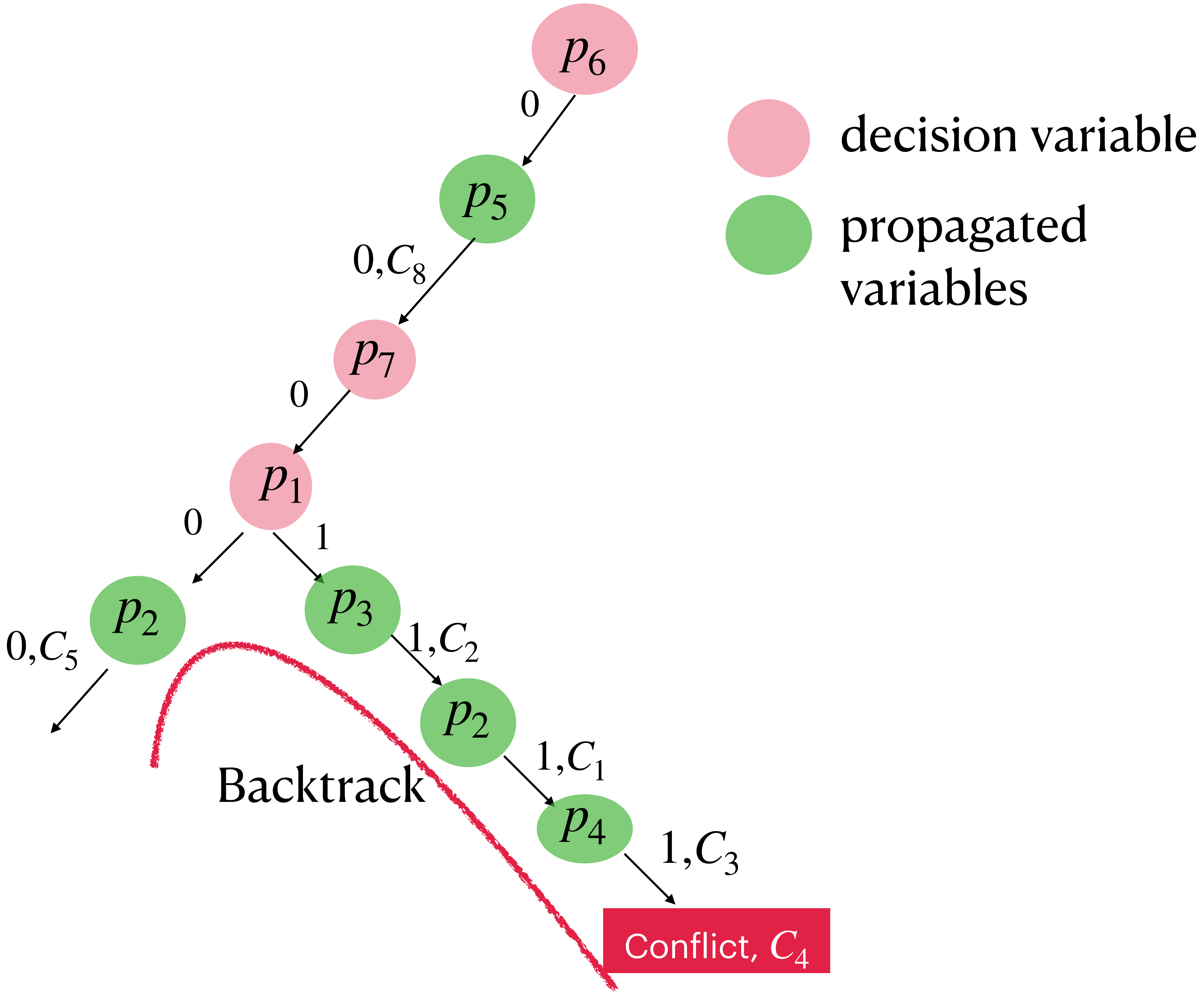

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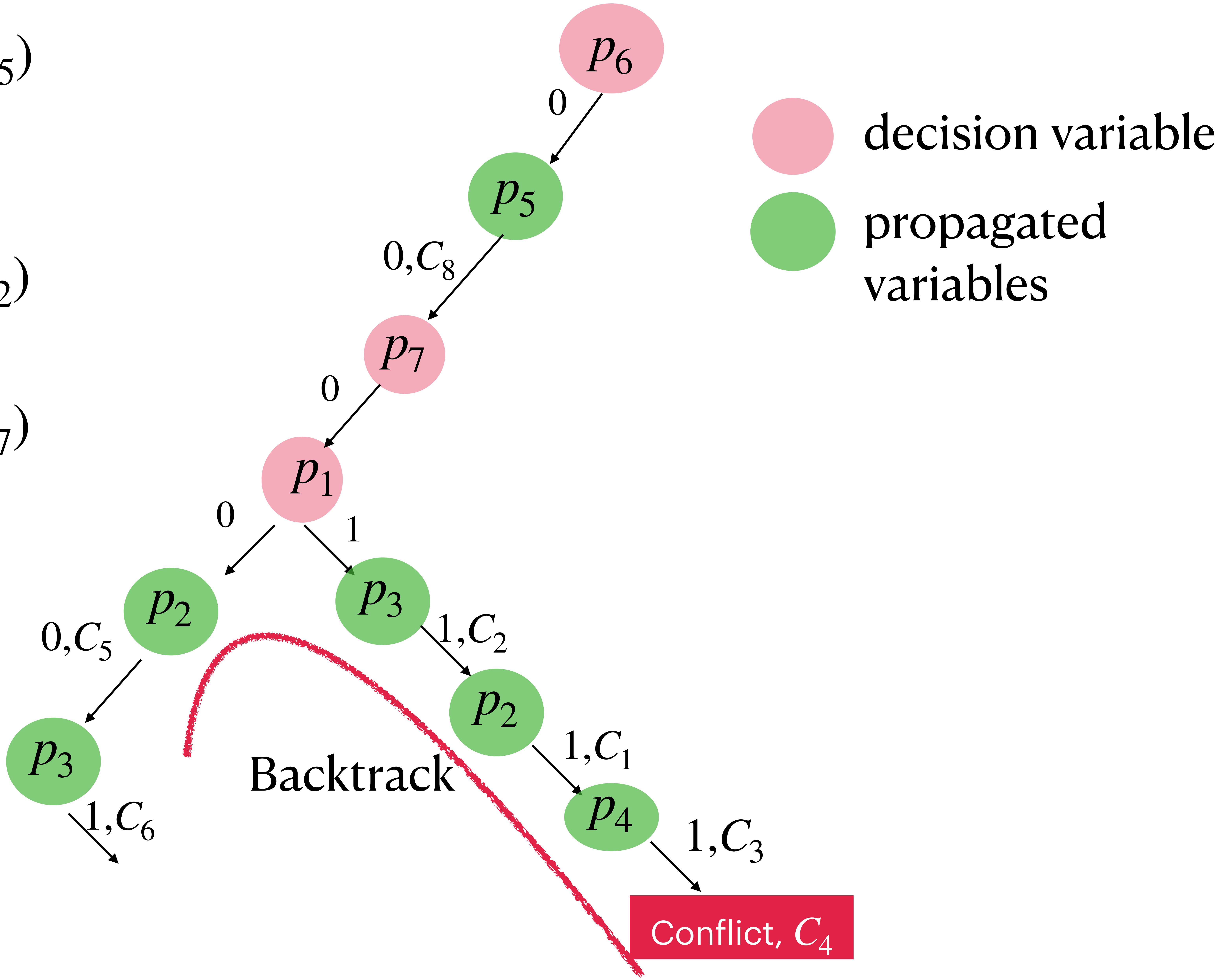

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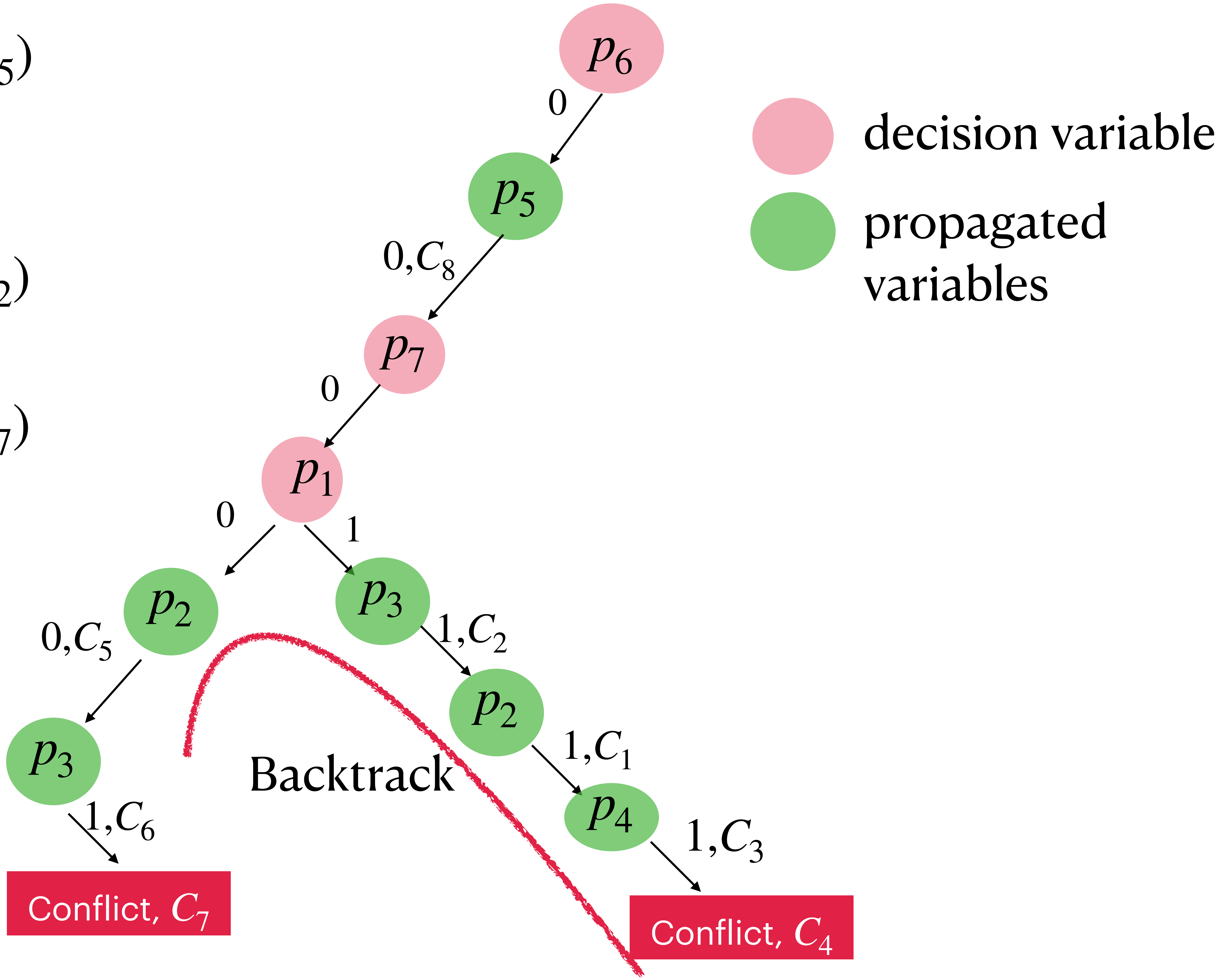

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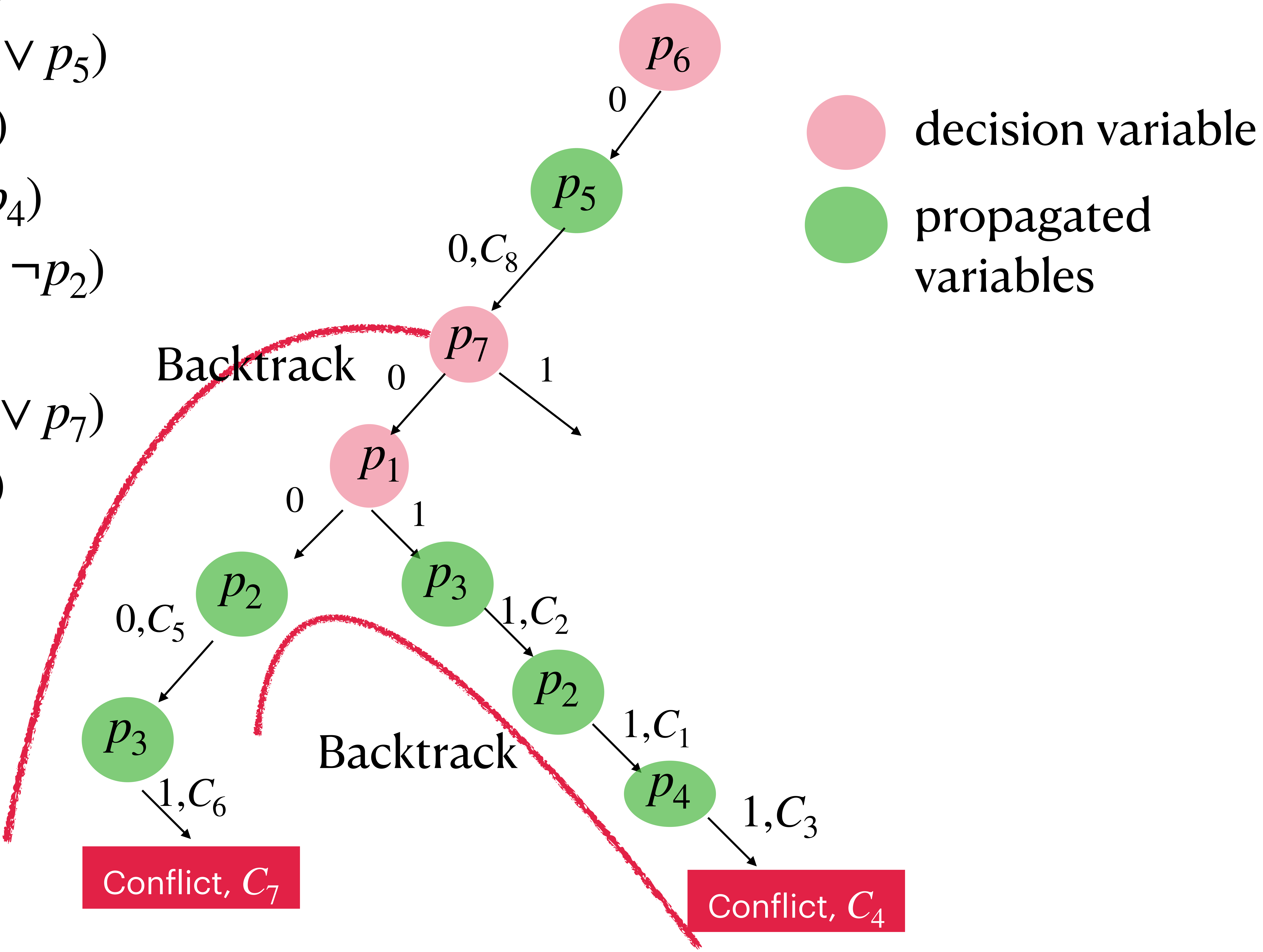

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An optimization of DPLL:

As we decide and propagate, we can observe the run, and avoid unnecessary backtracking.

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Implication Graph.

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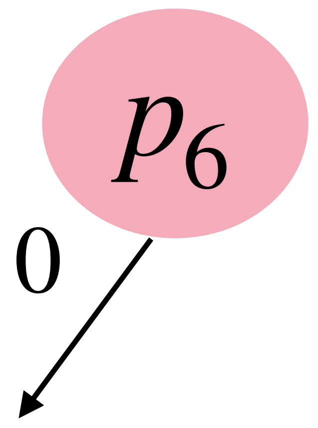

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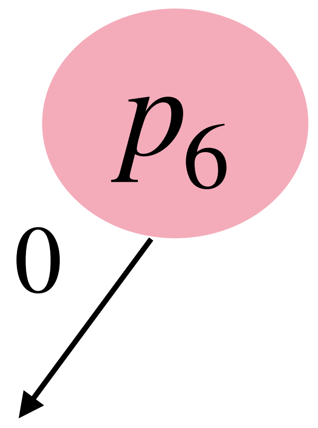

$\neg p_6 @ 1$

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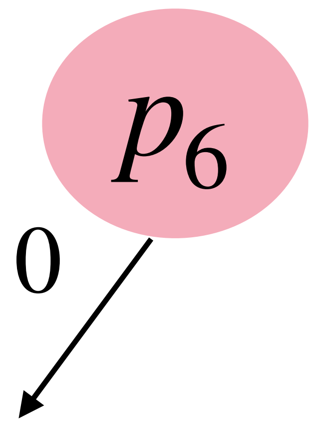

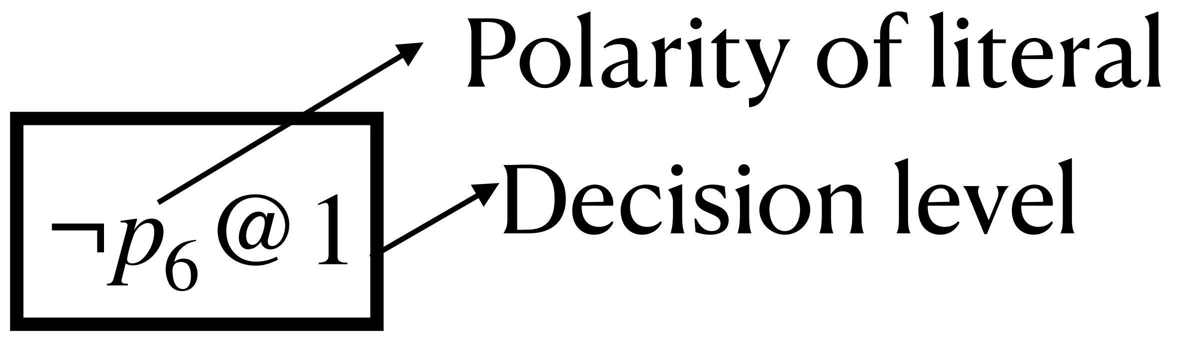

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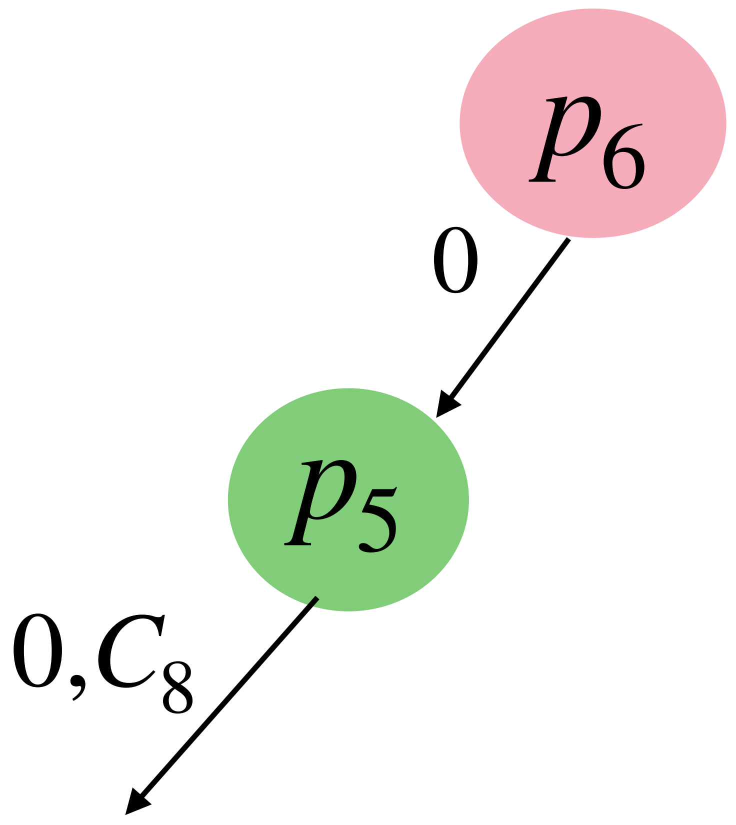

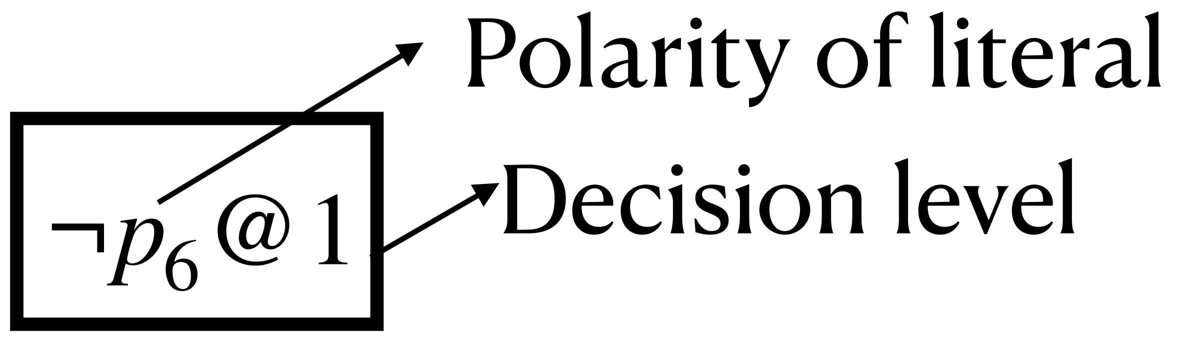

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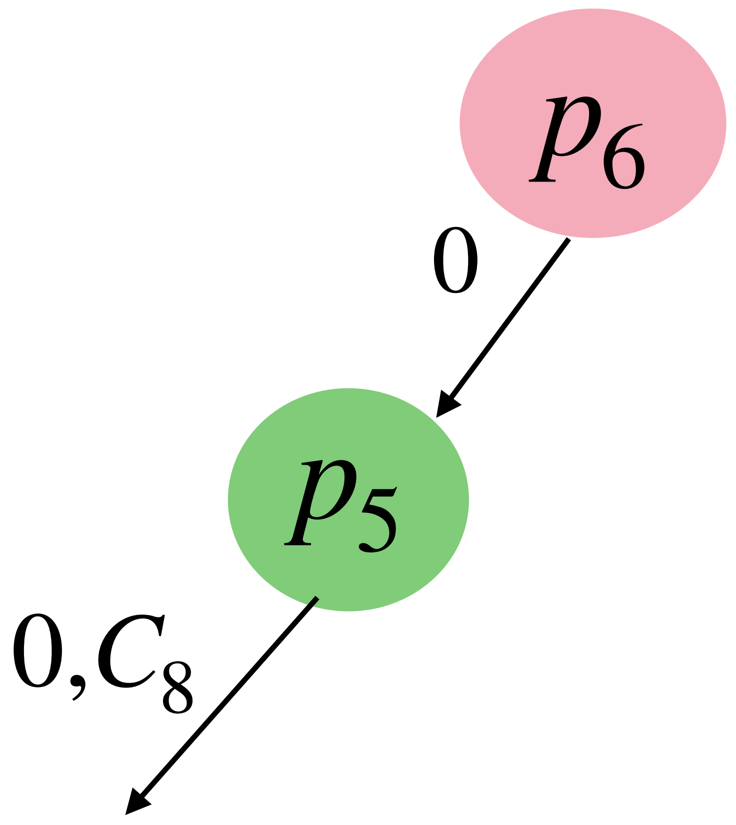

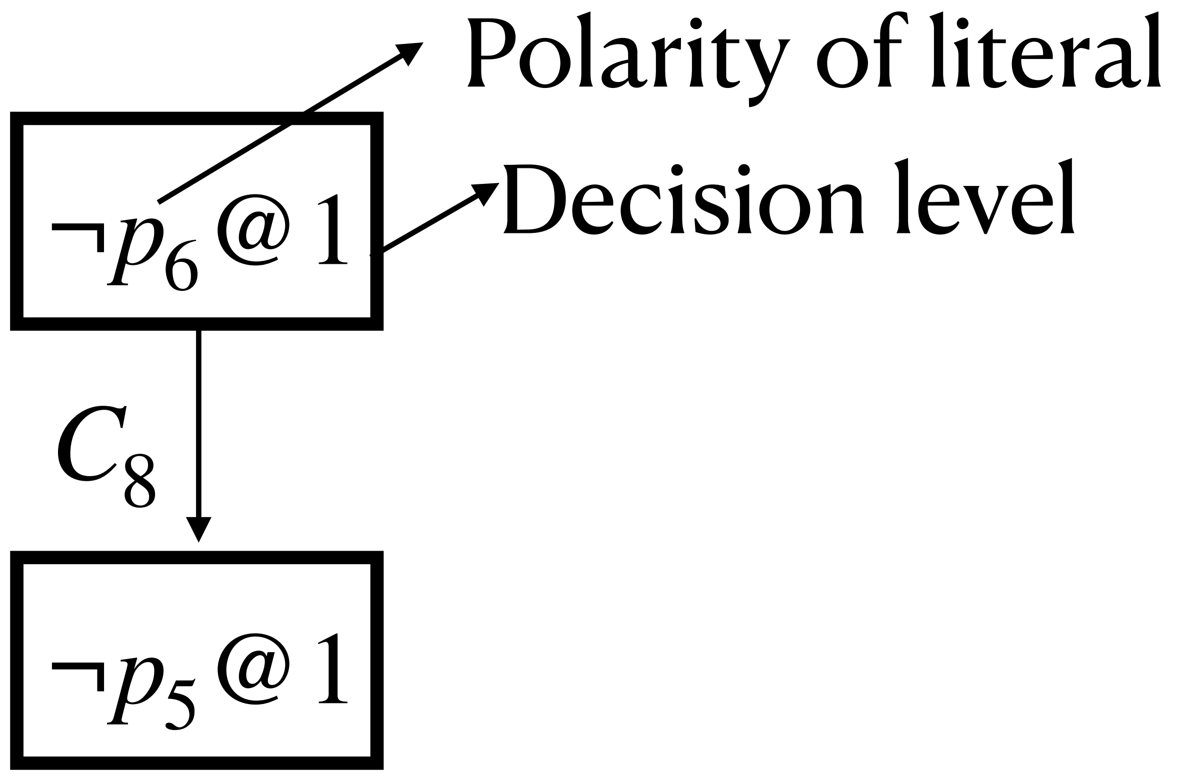

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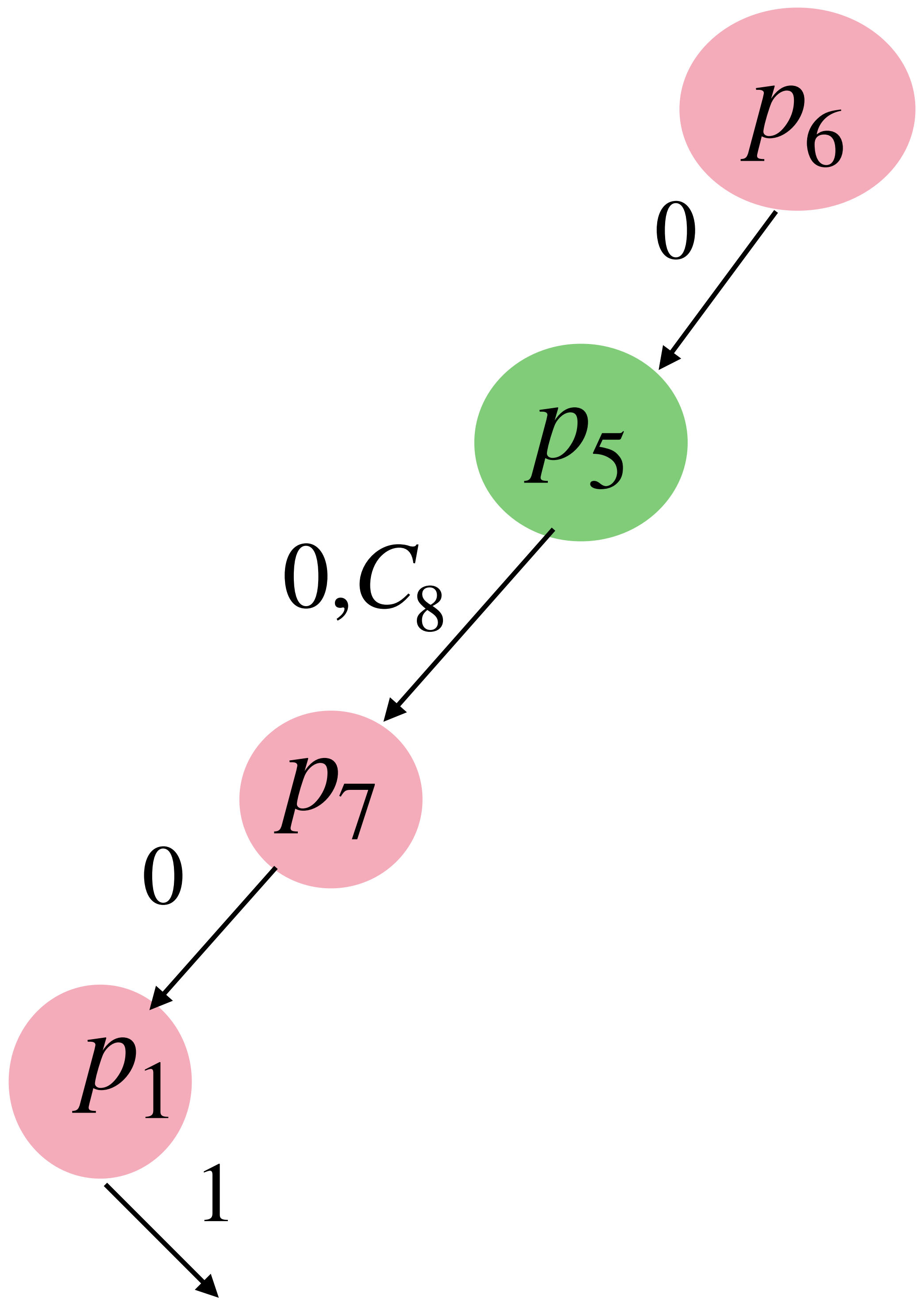

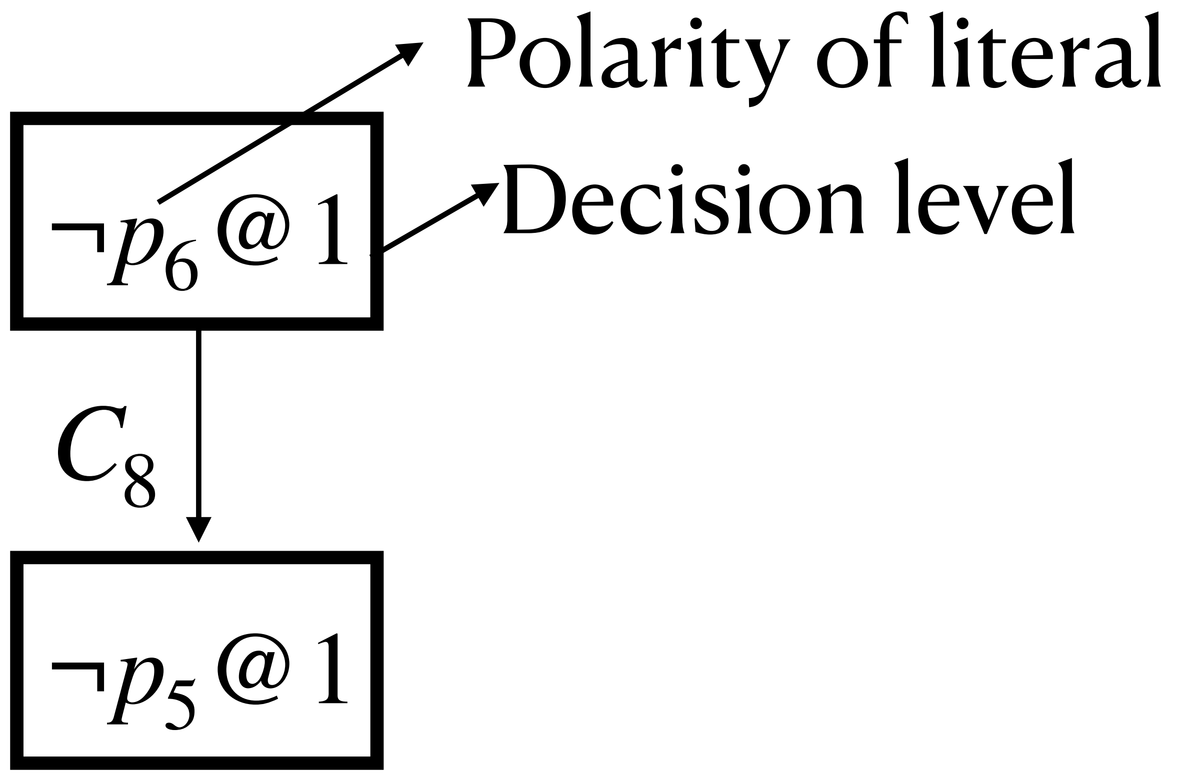

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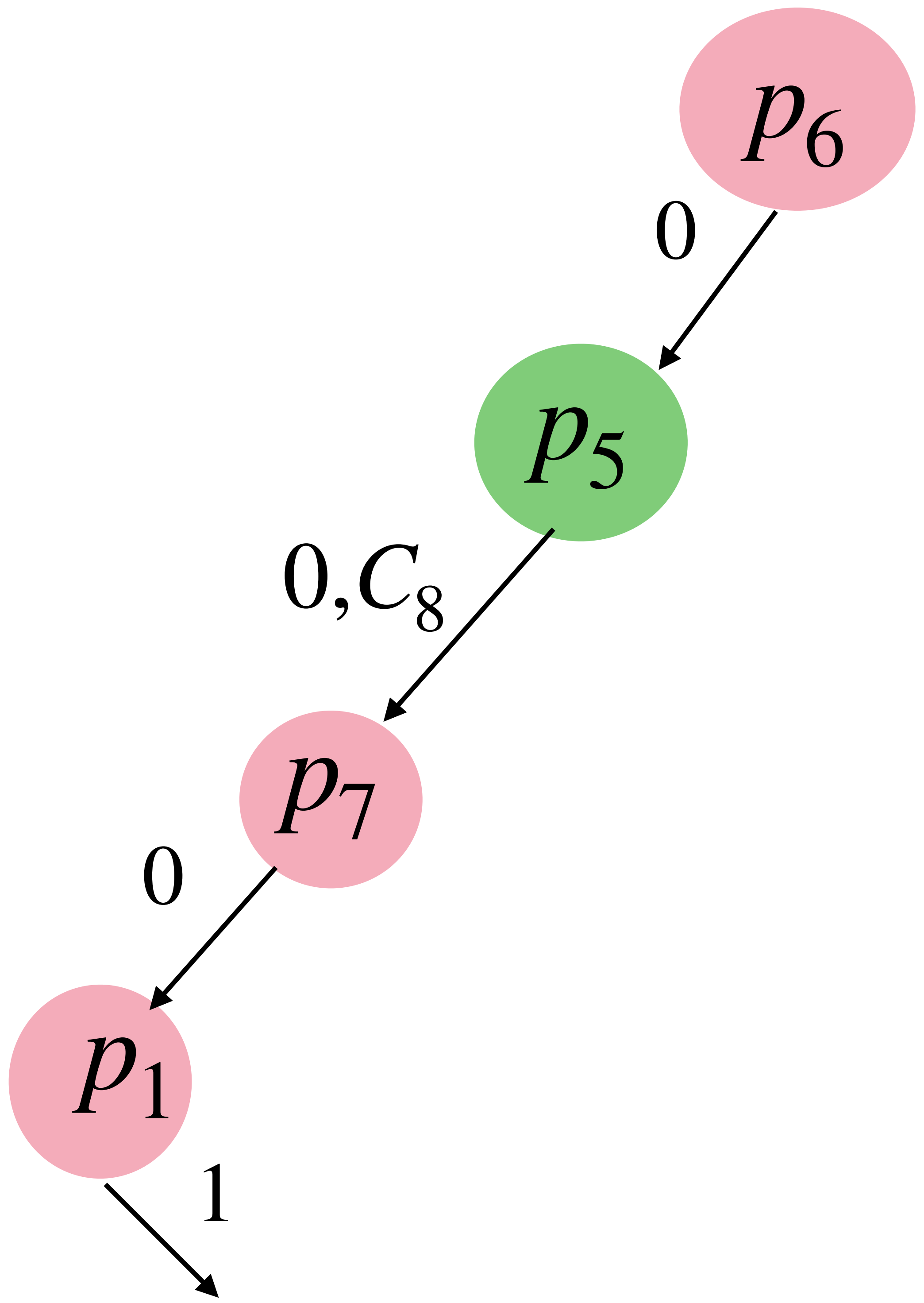

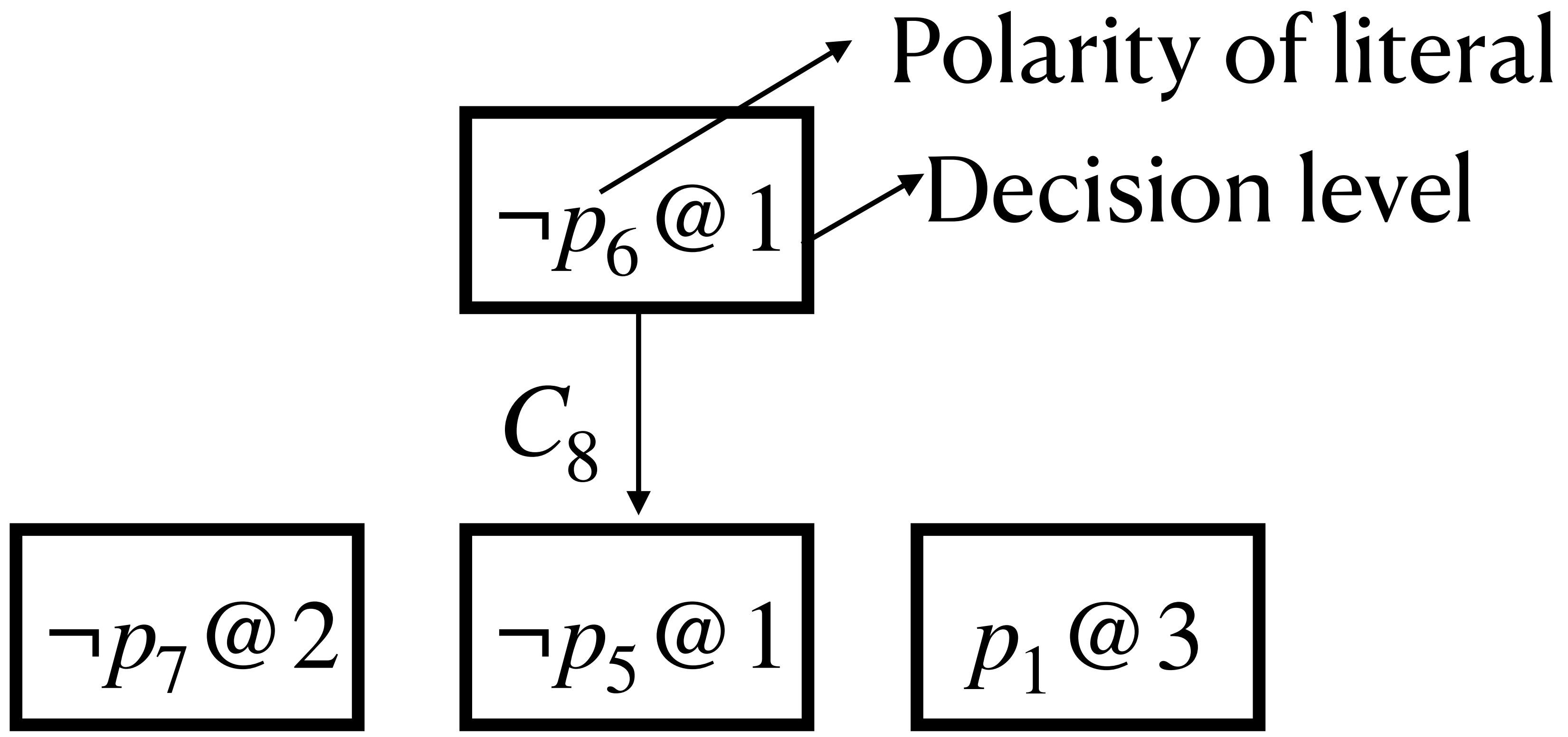

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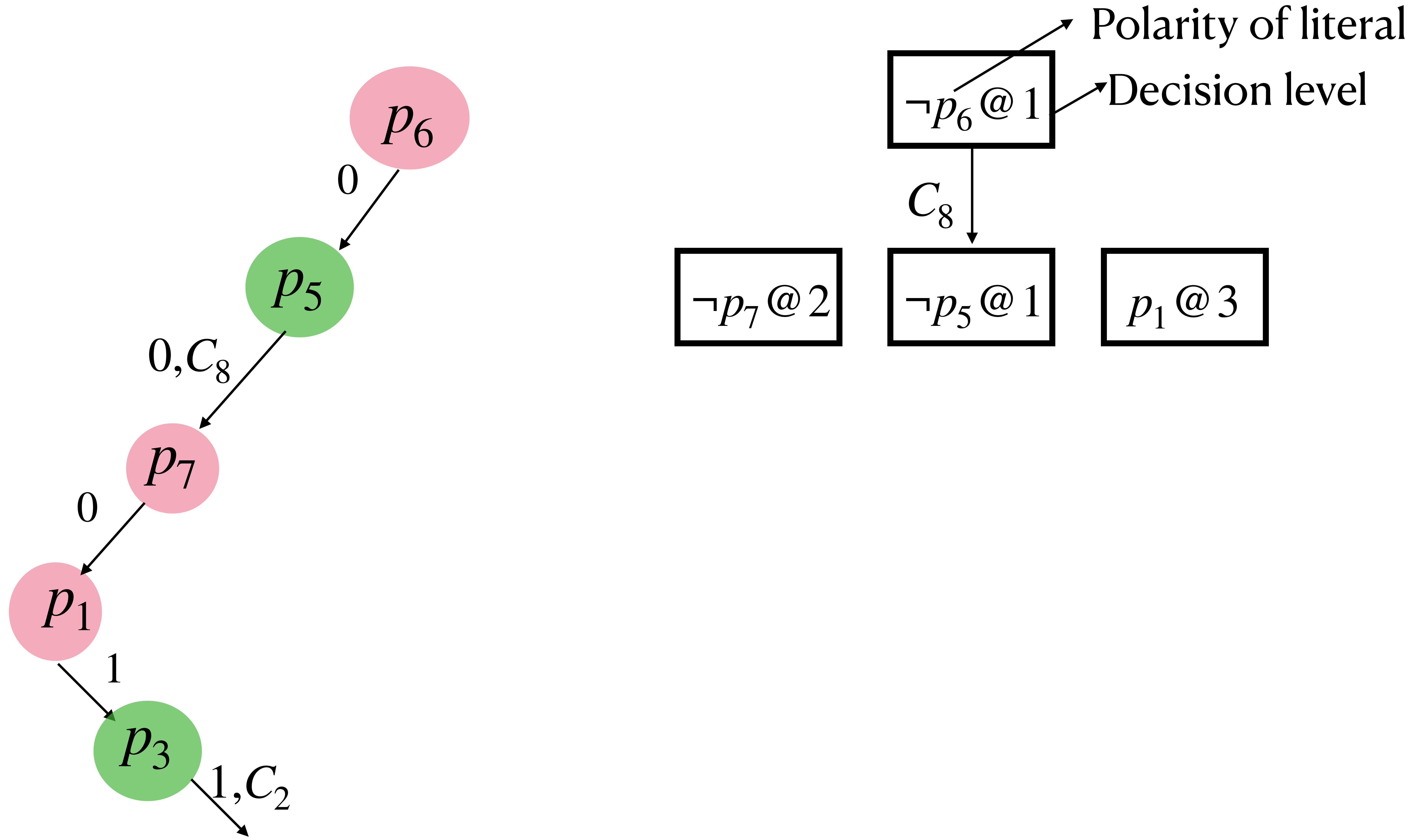

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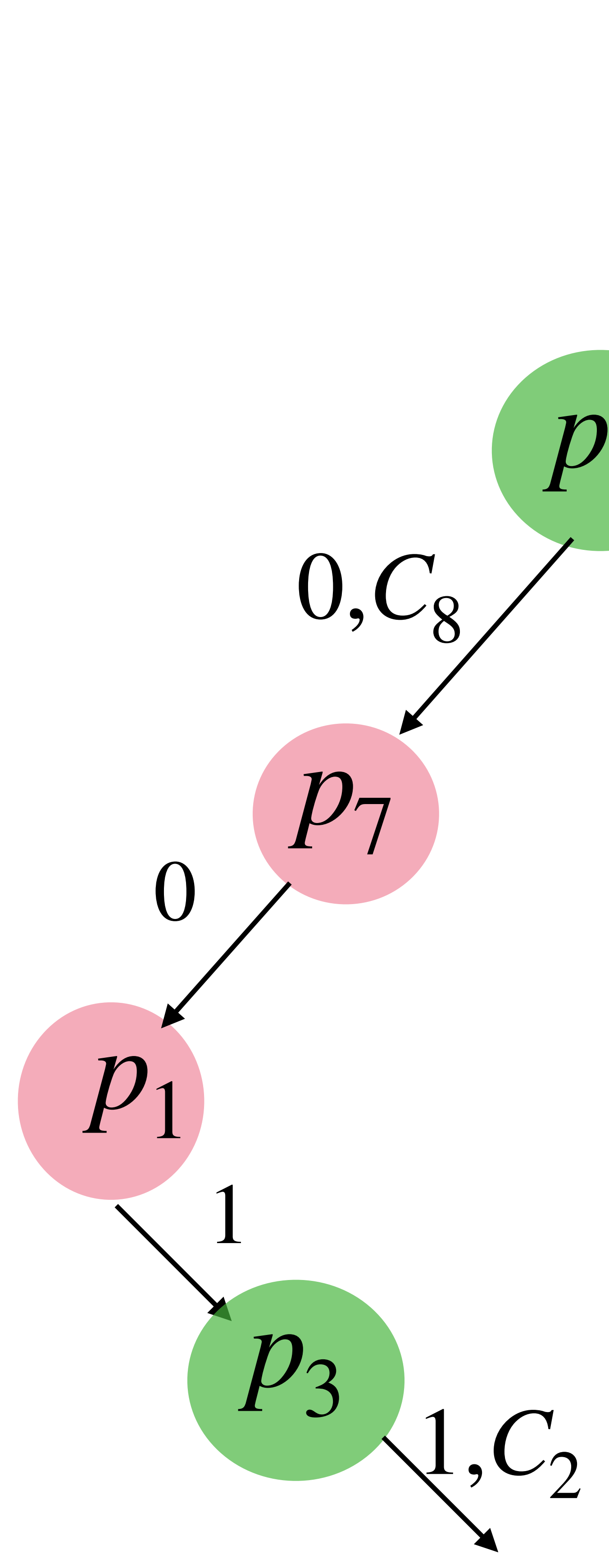

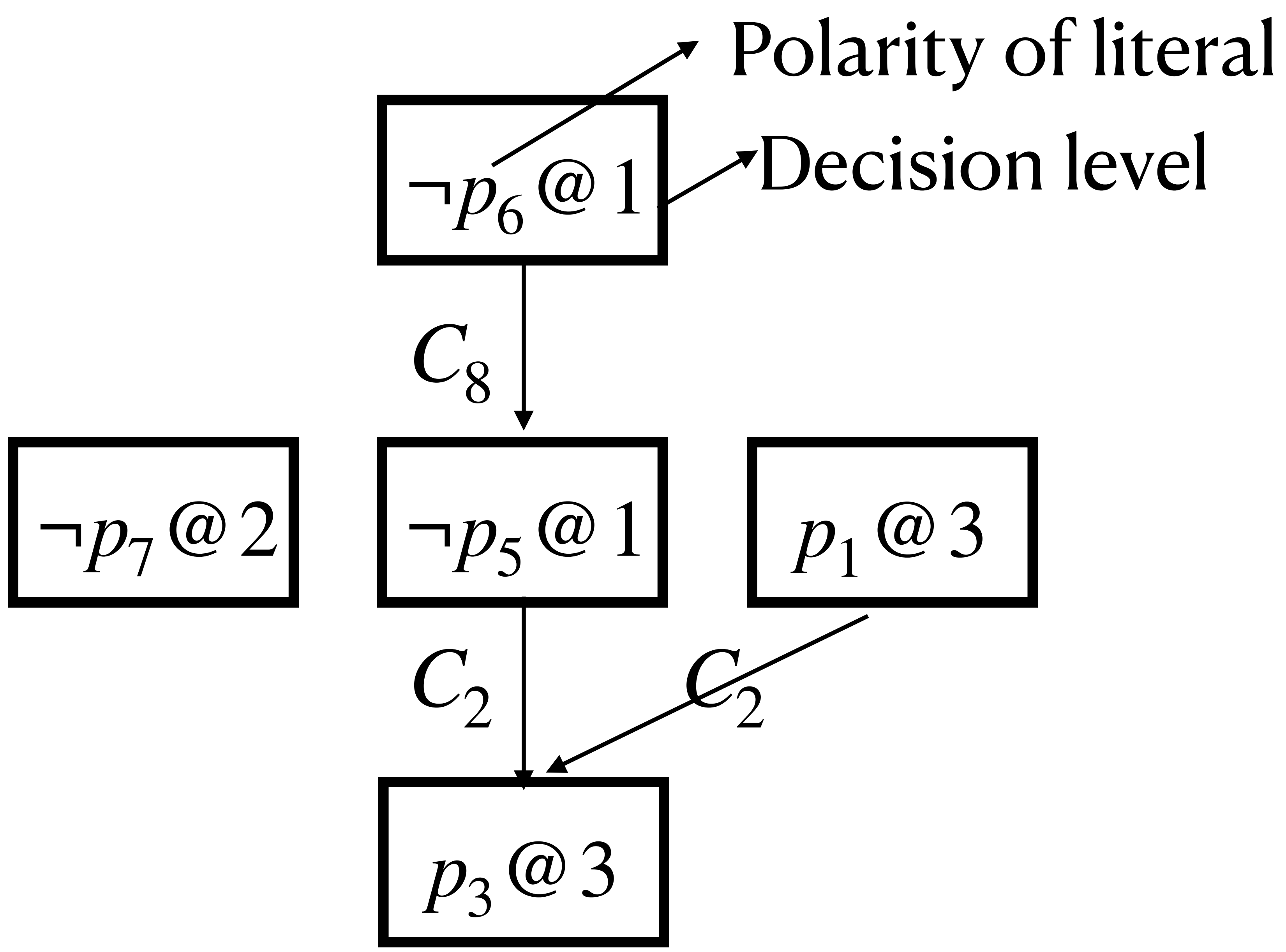

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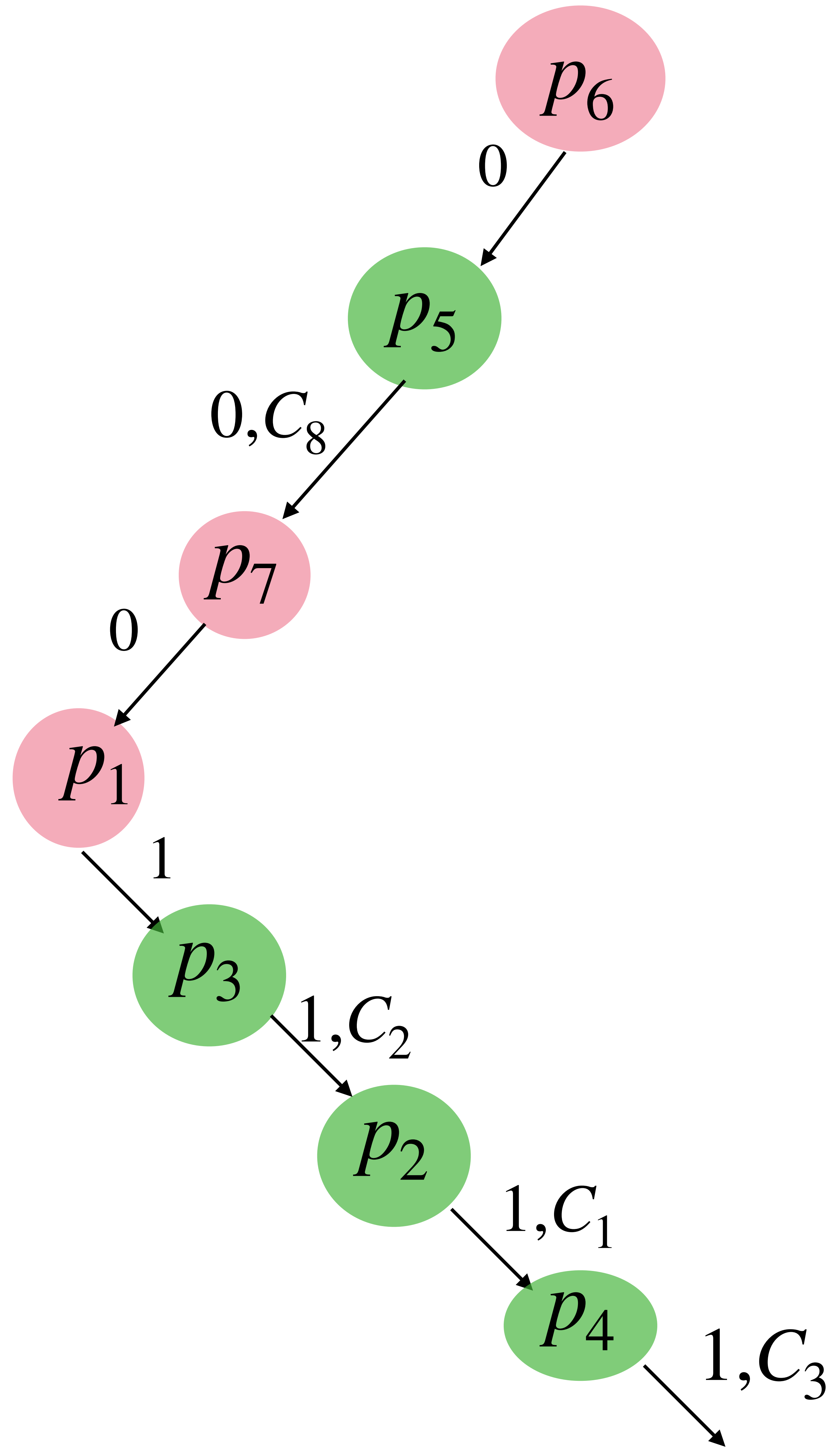

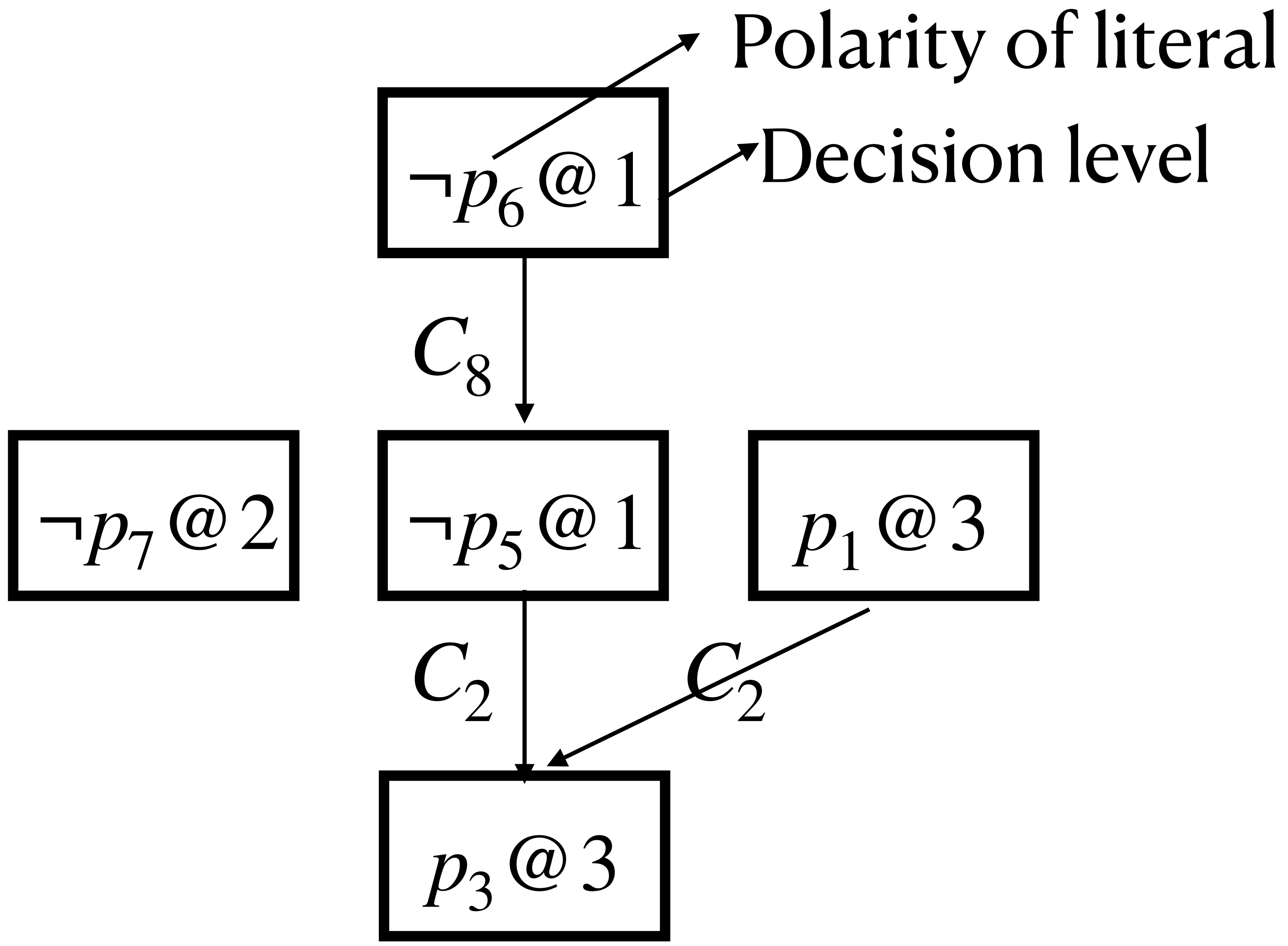

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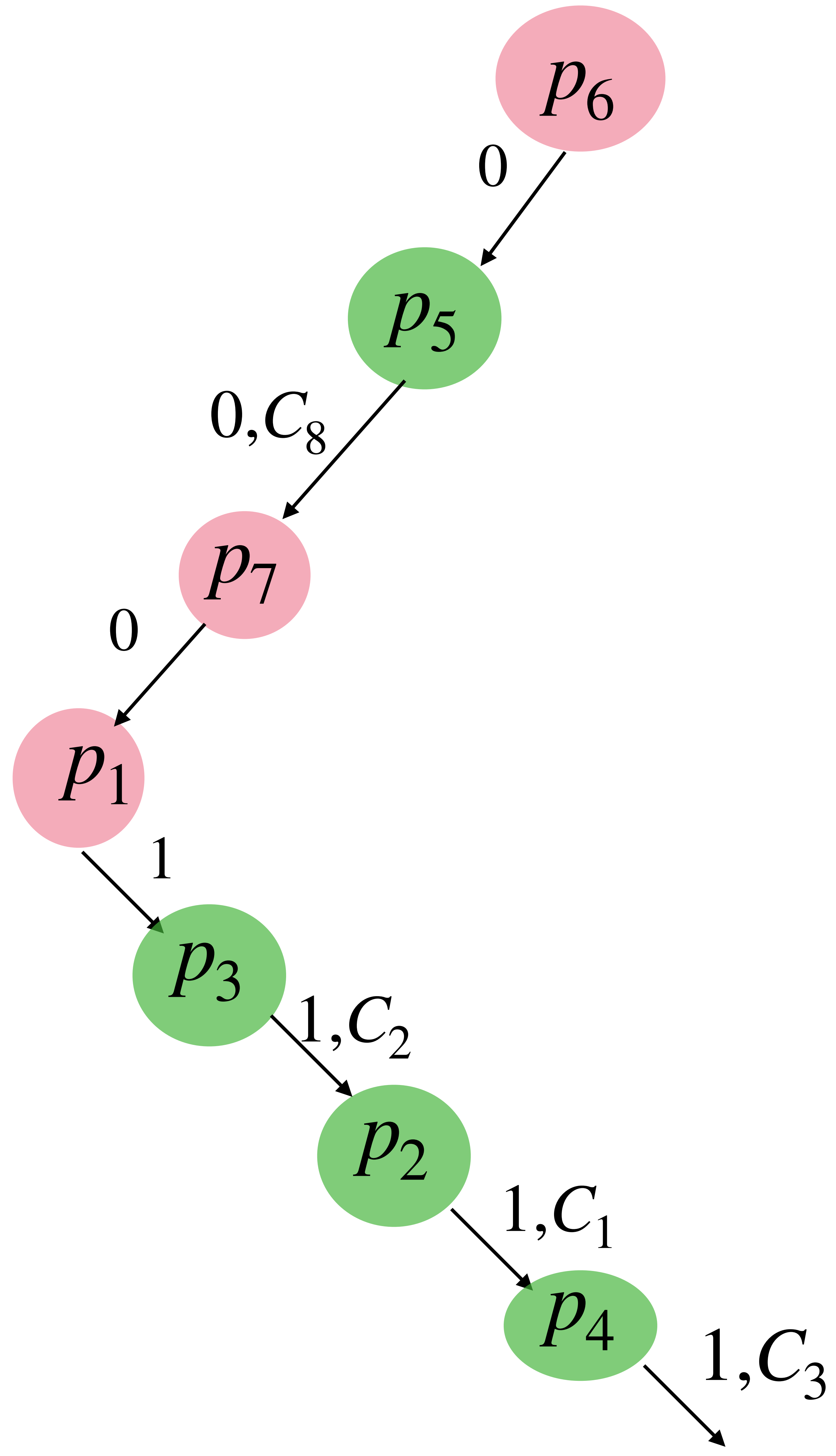

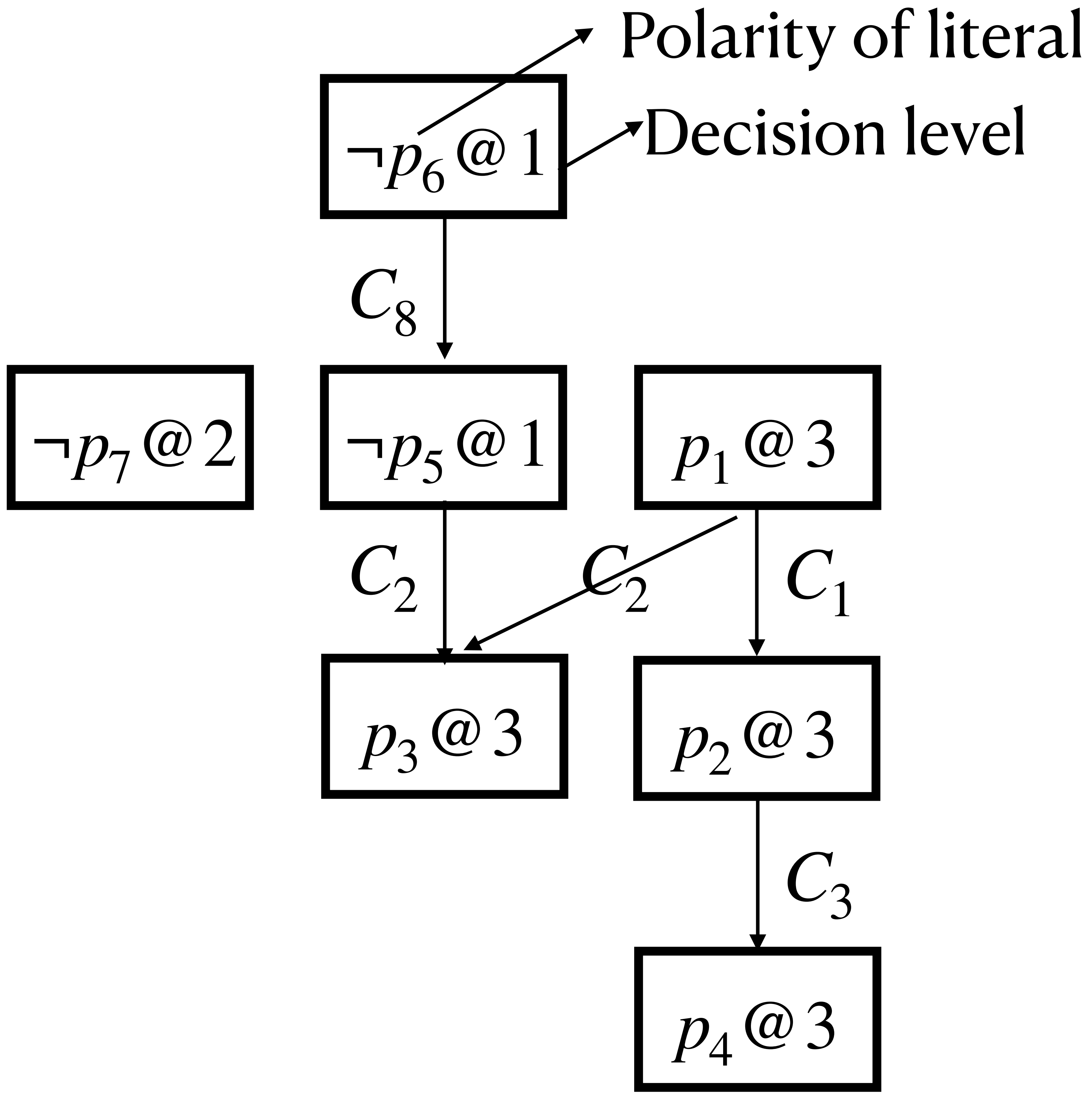

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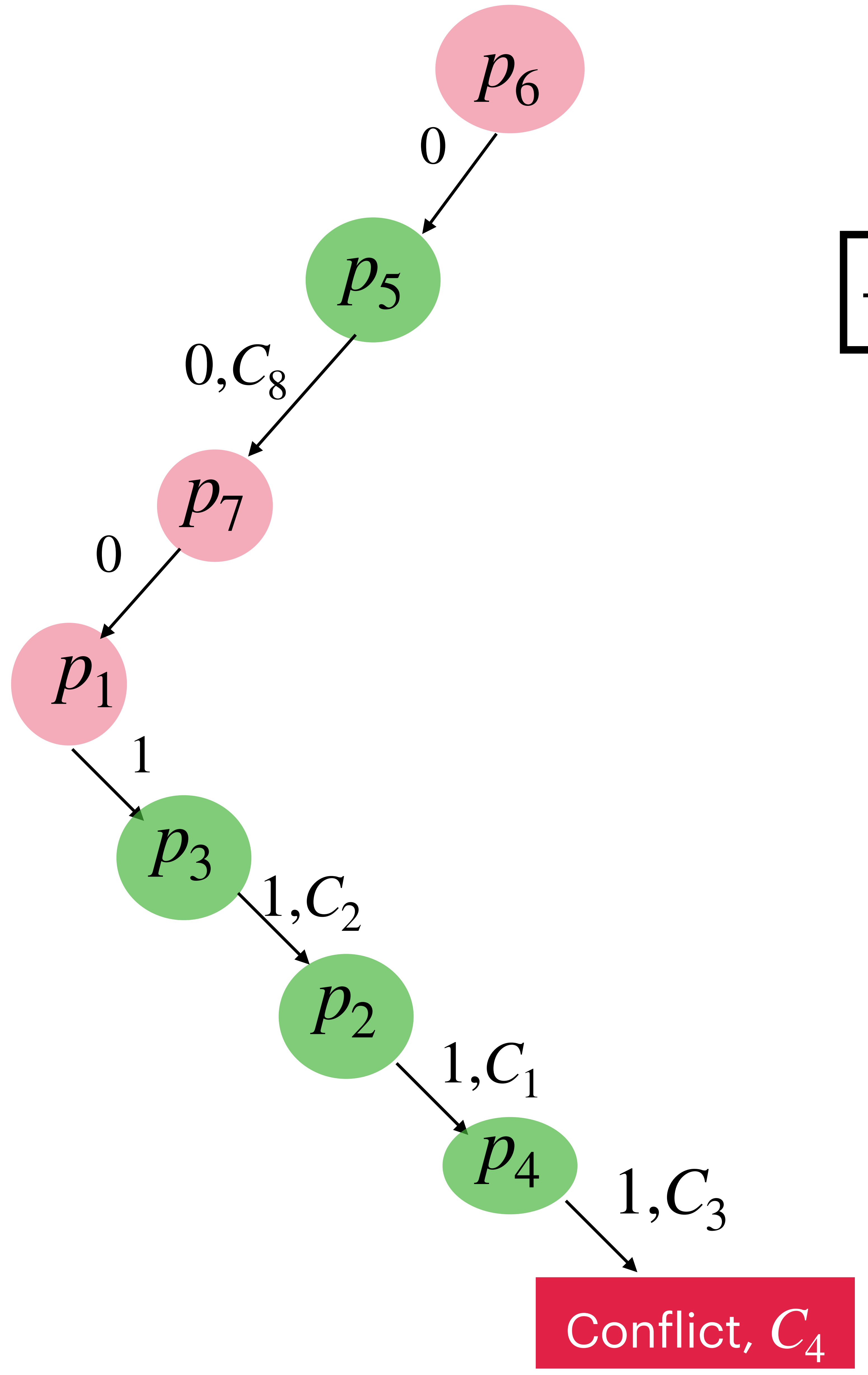

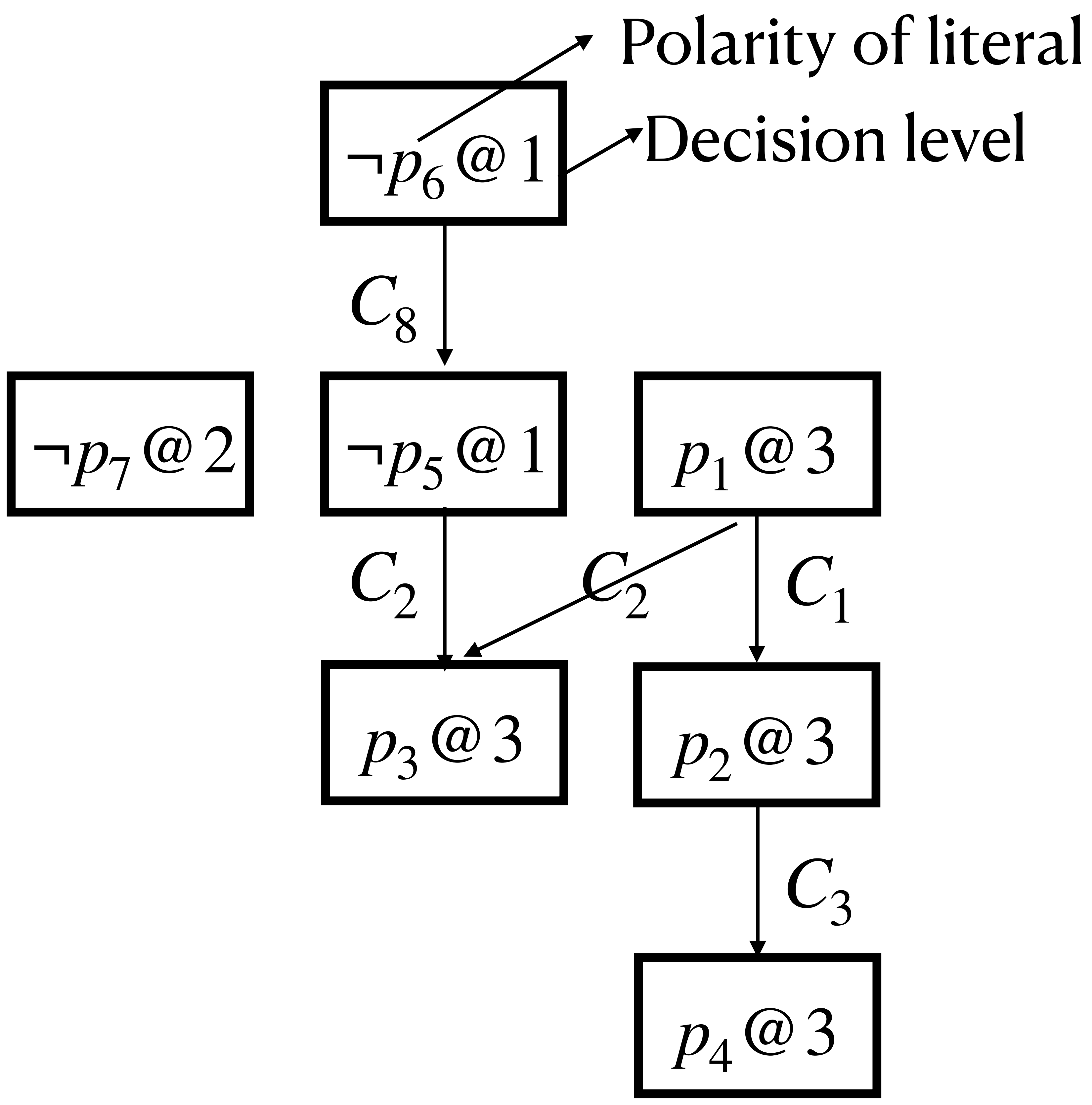

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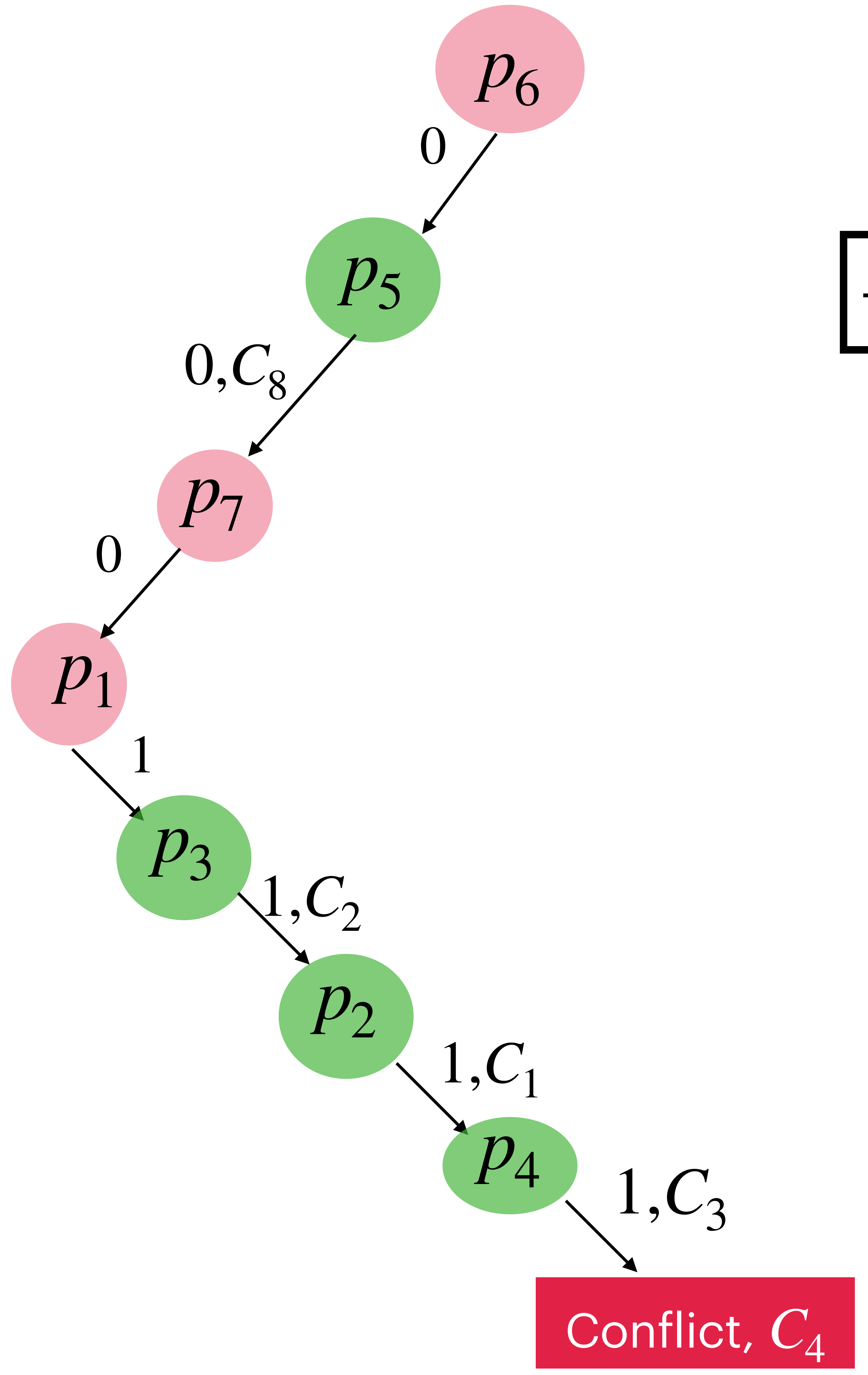

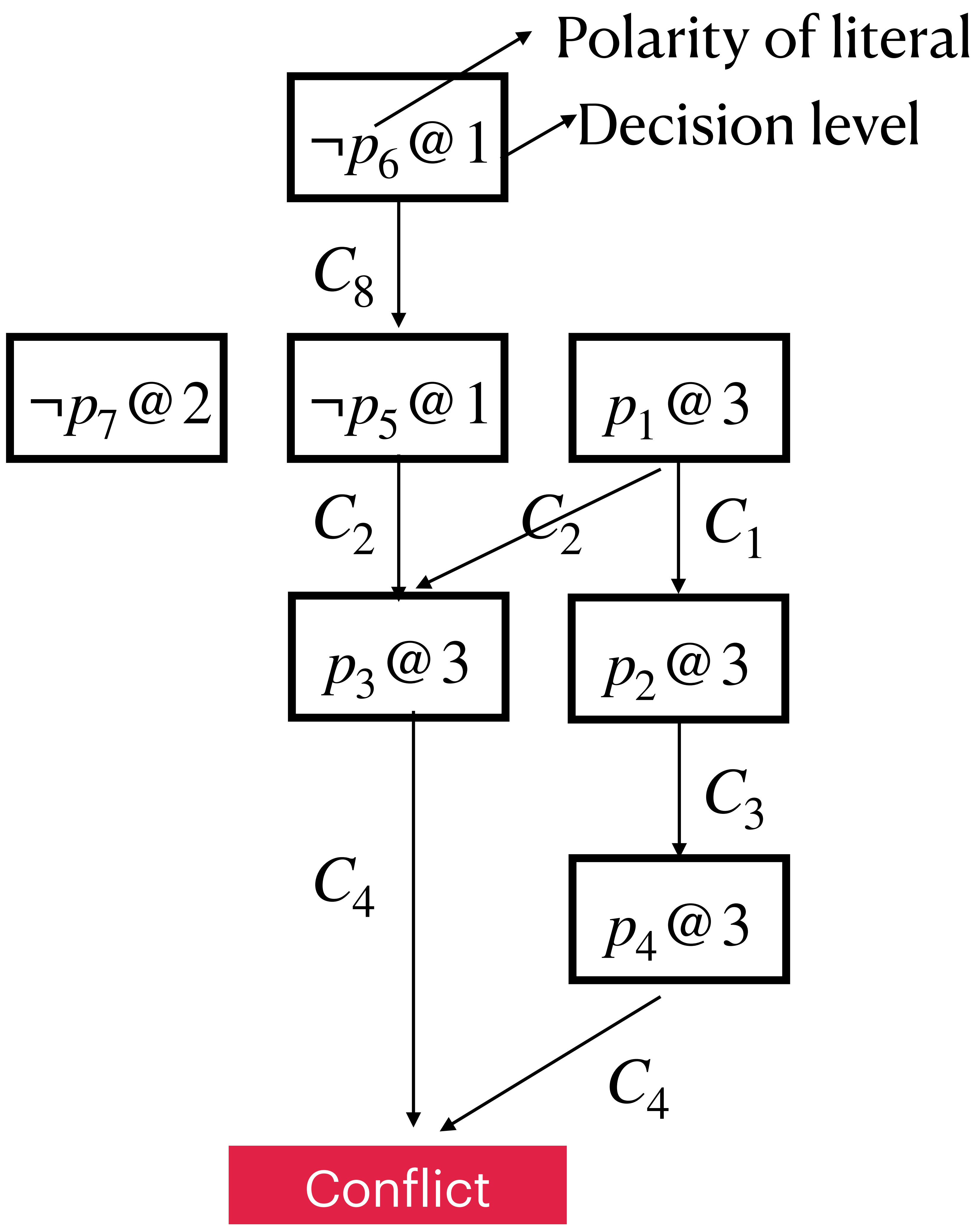

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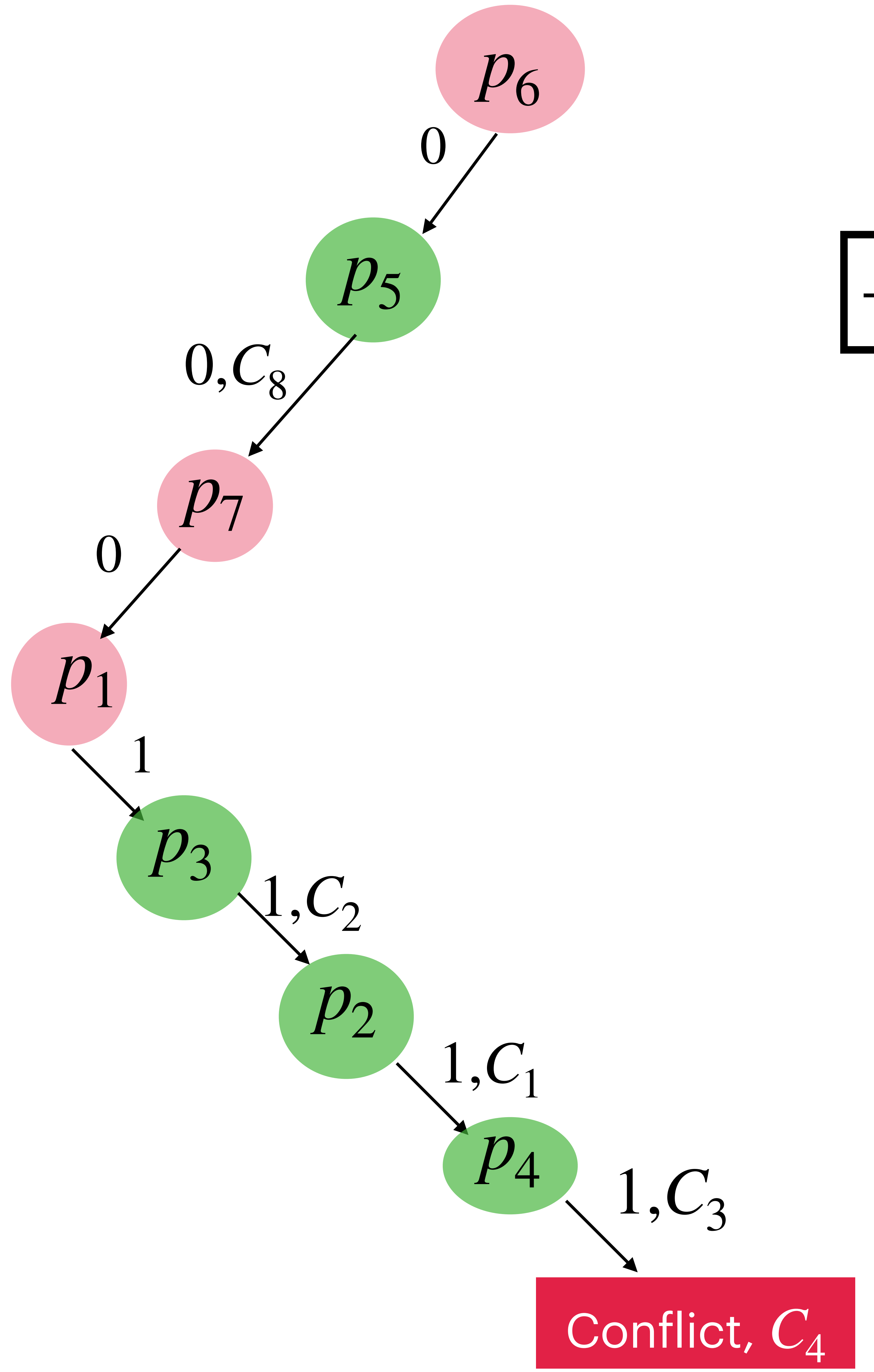
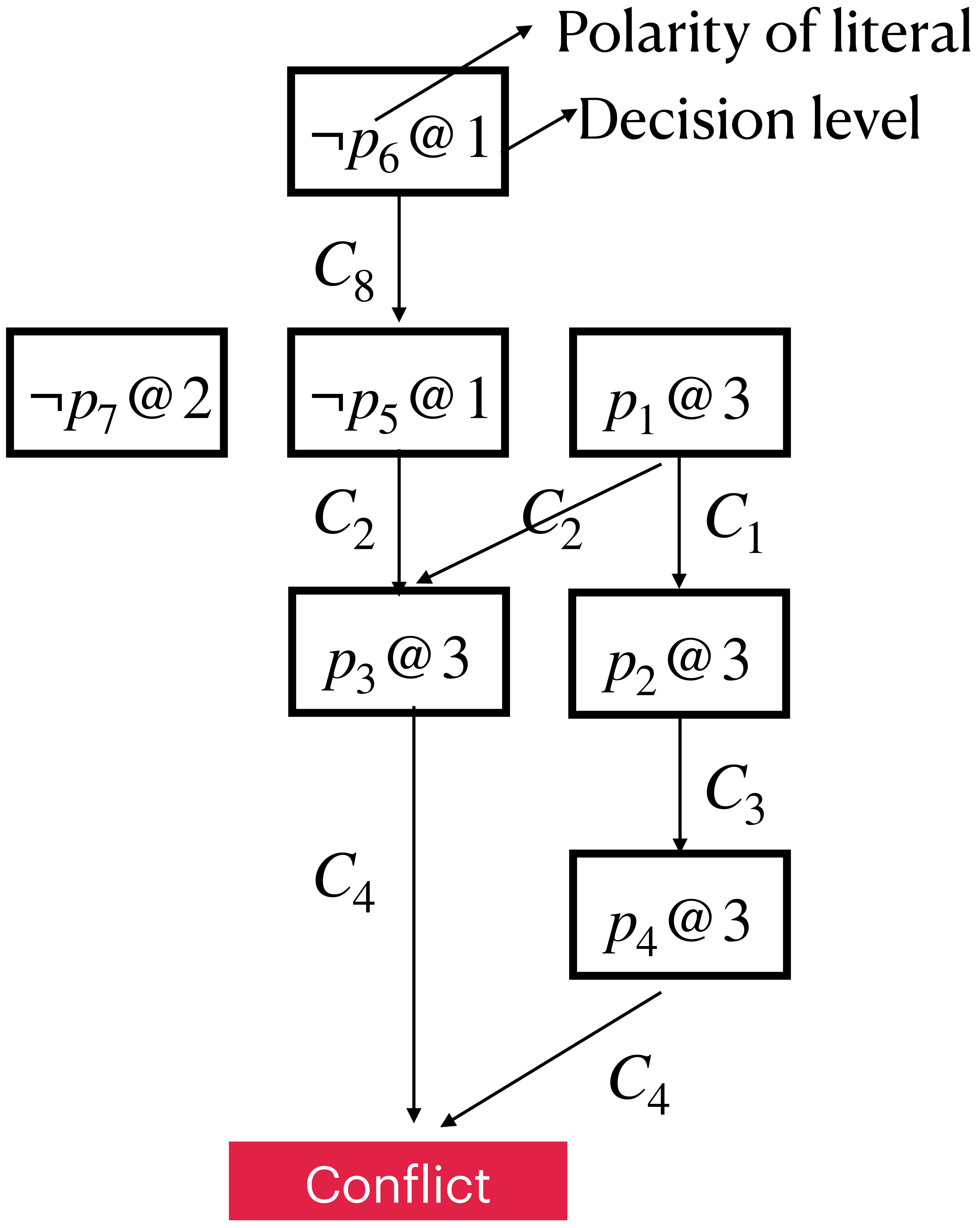
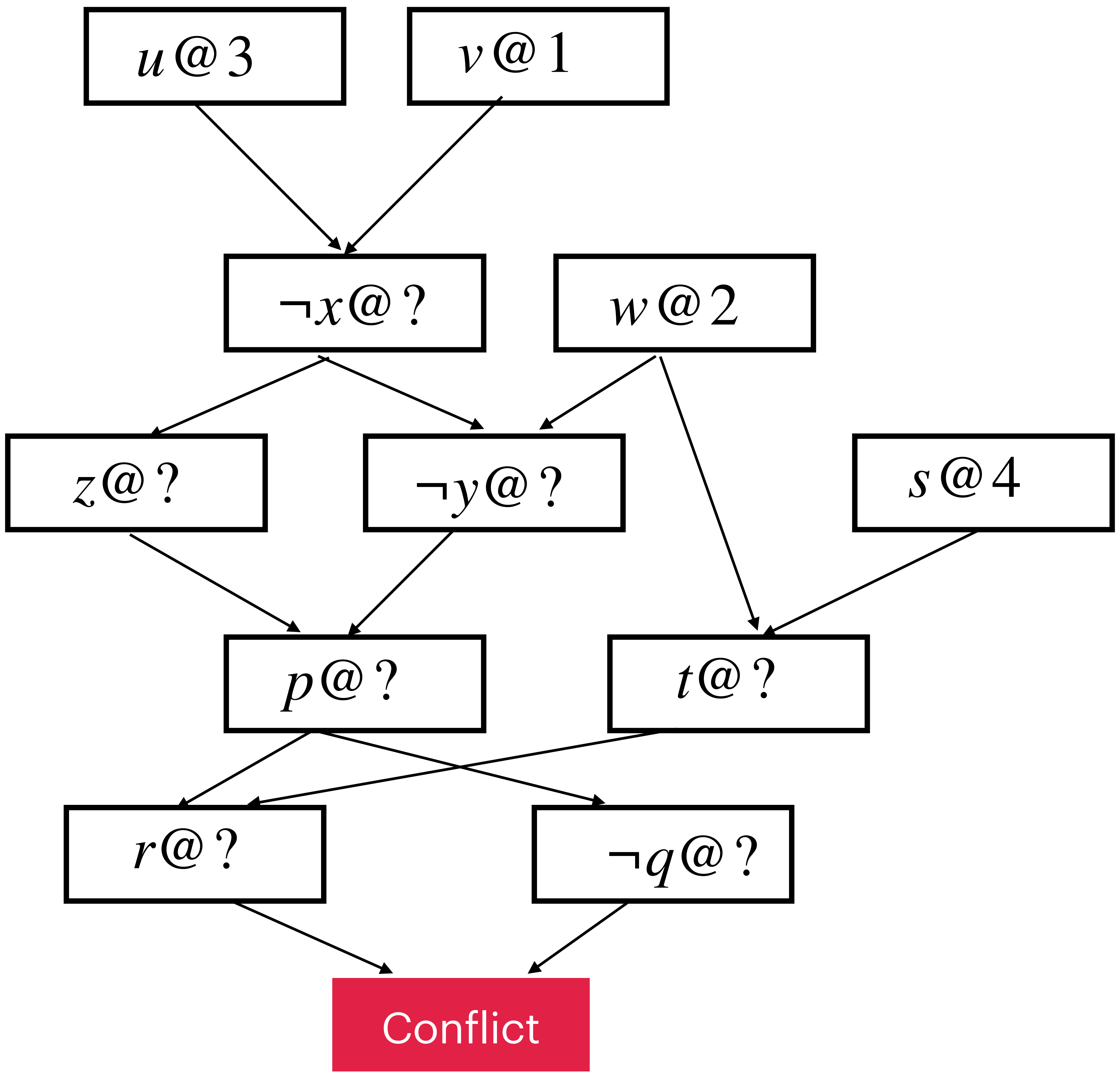




Implication Graph.

Assign decision level at every node in implication graph



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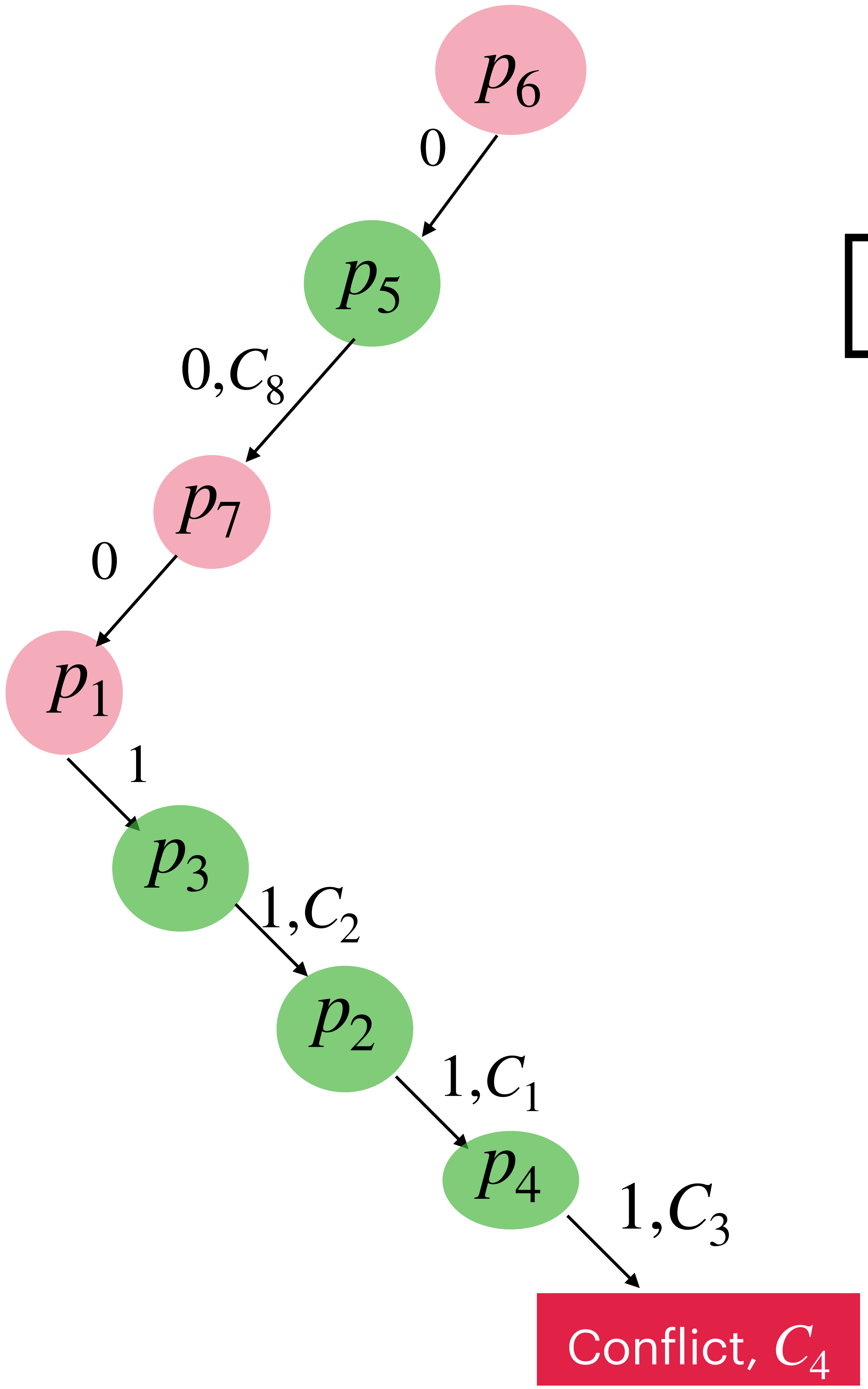
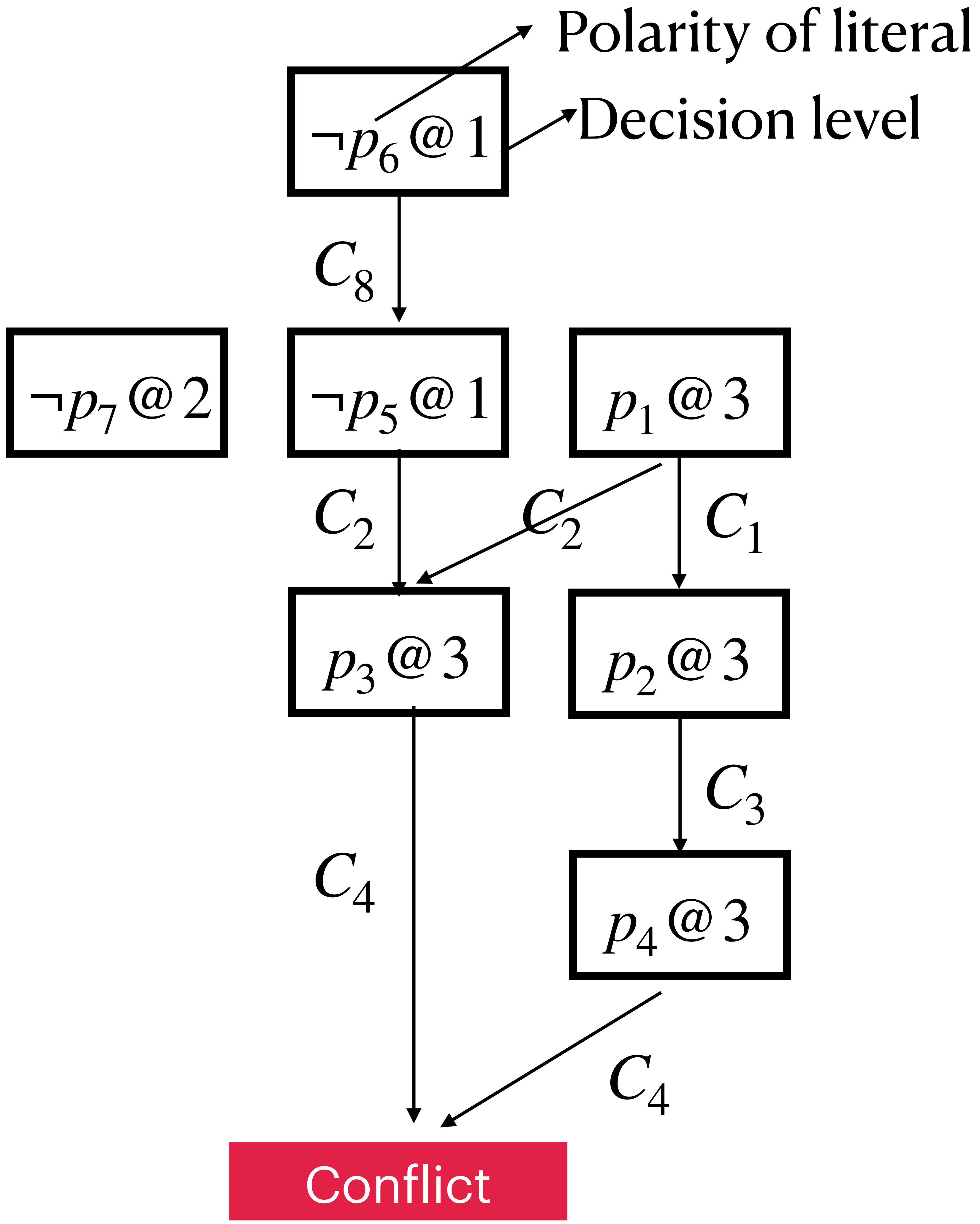
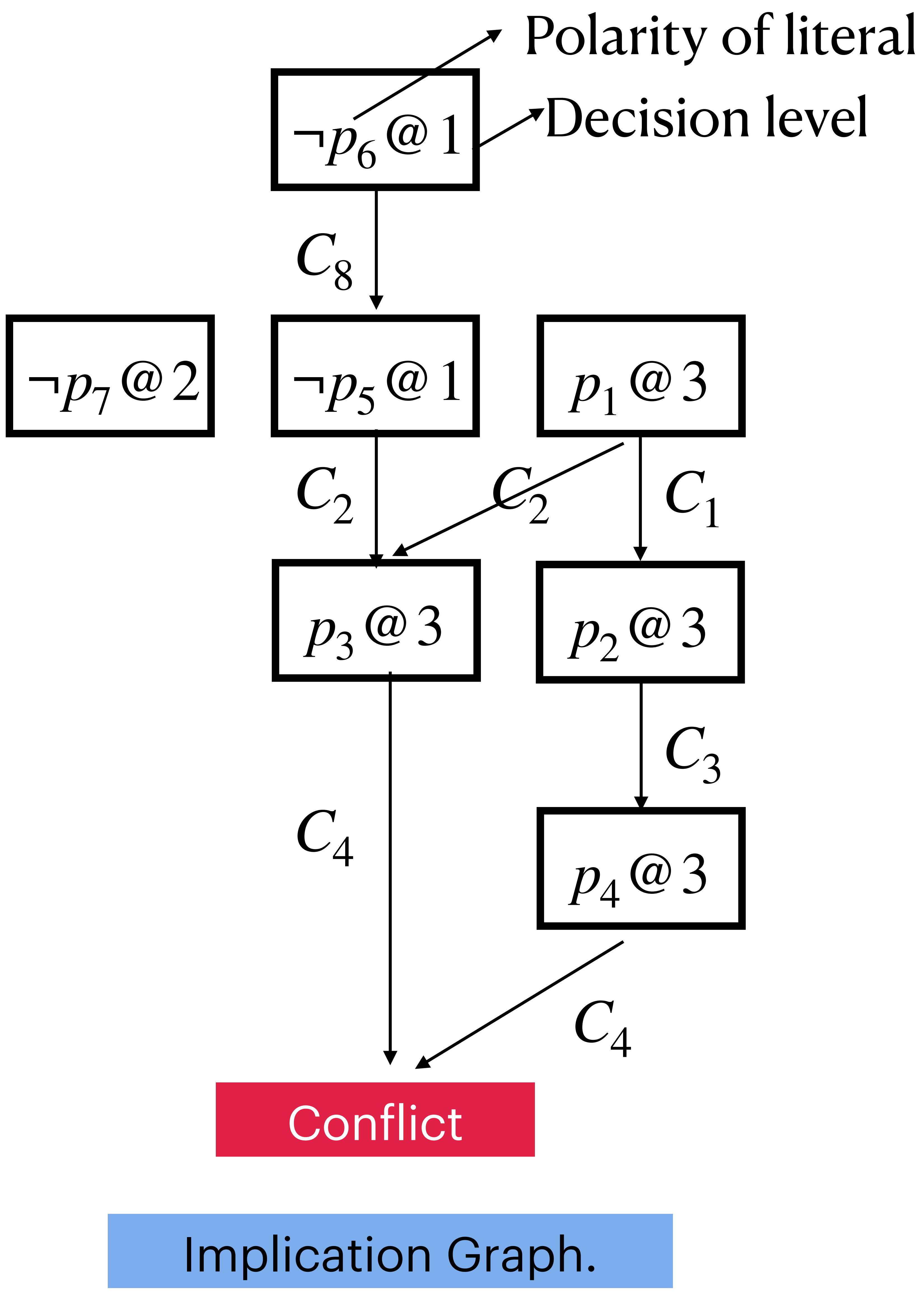




Implication Graph.

Conflict Clause

The clause of the negations of the causing decisions is called conflict clause.



Conflict Clause

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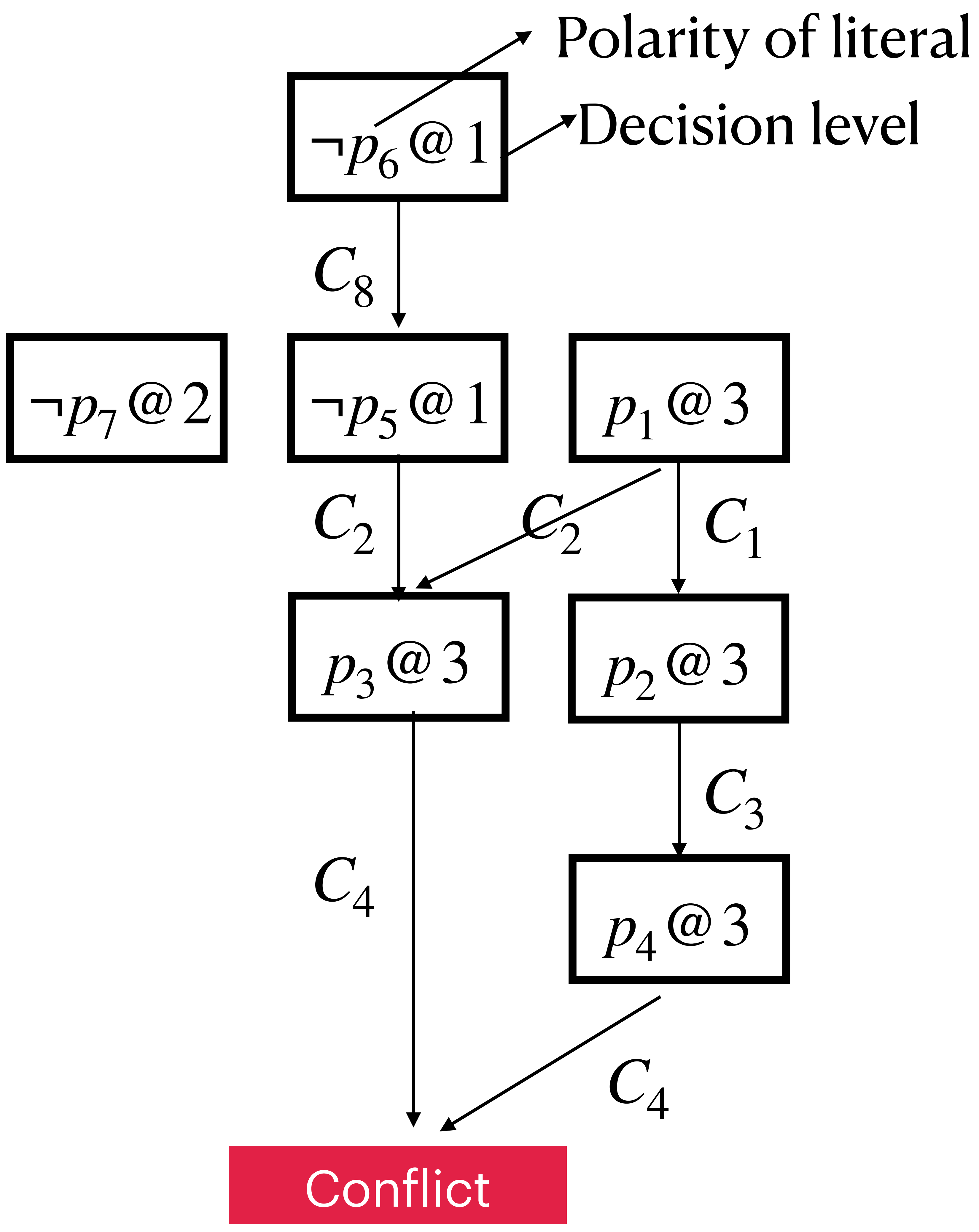
Mistake: $p_6 = 0$ and $p_1 = 1$



Implication Graph.

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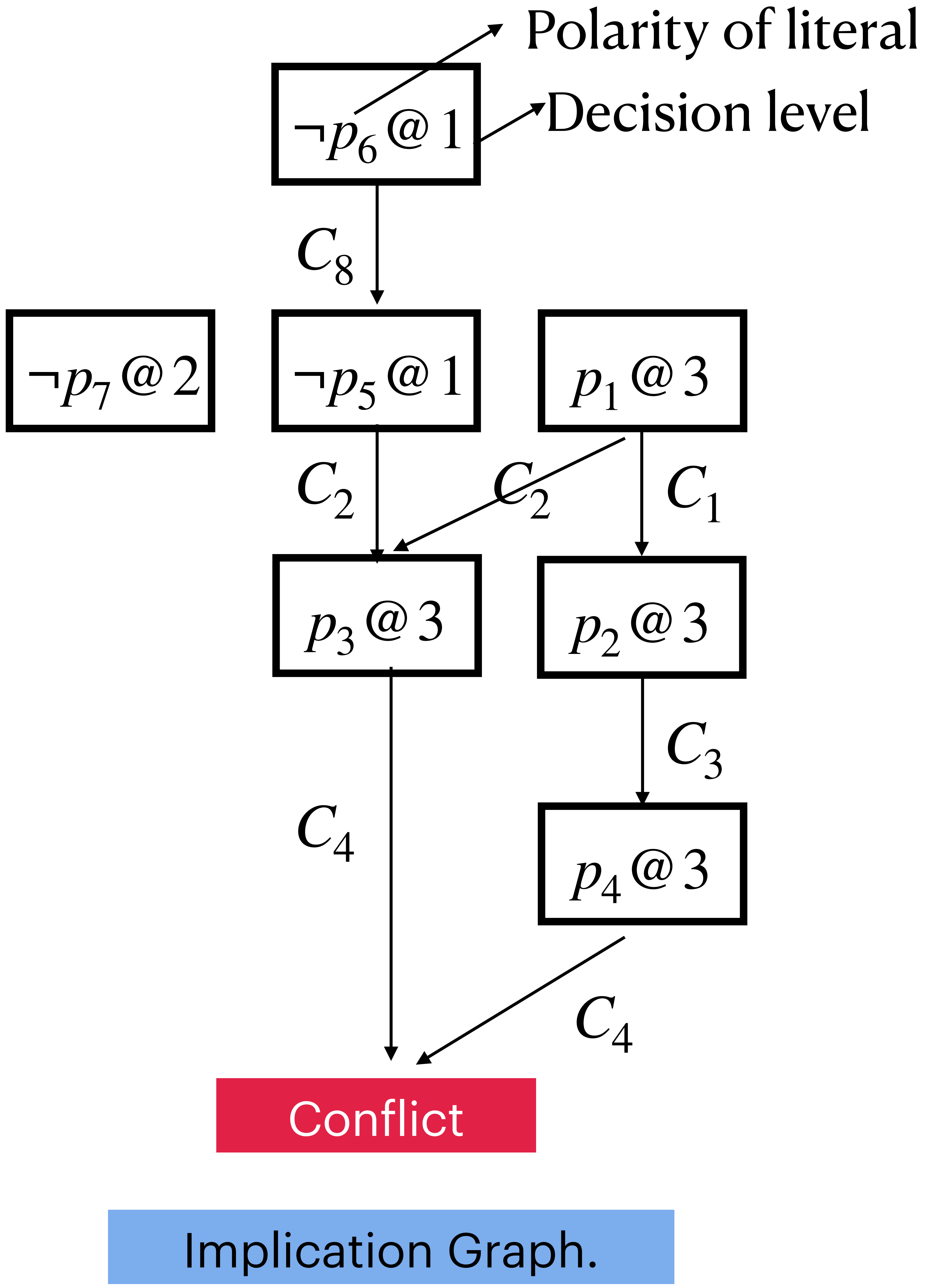

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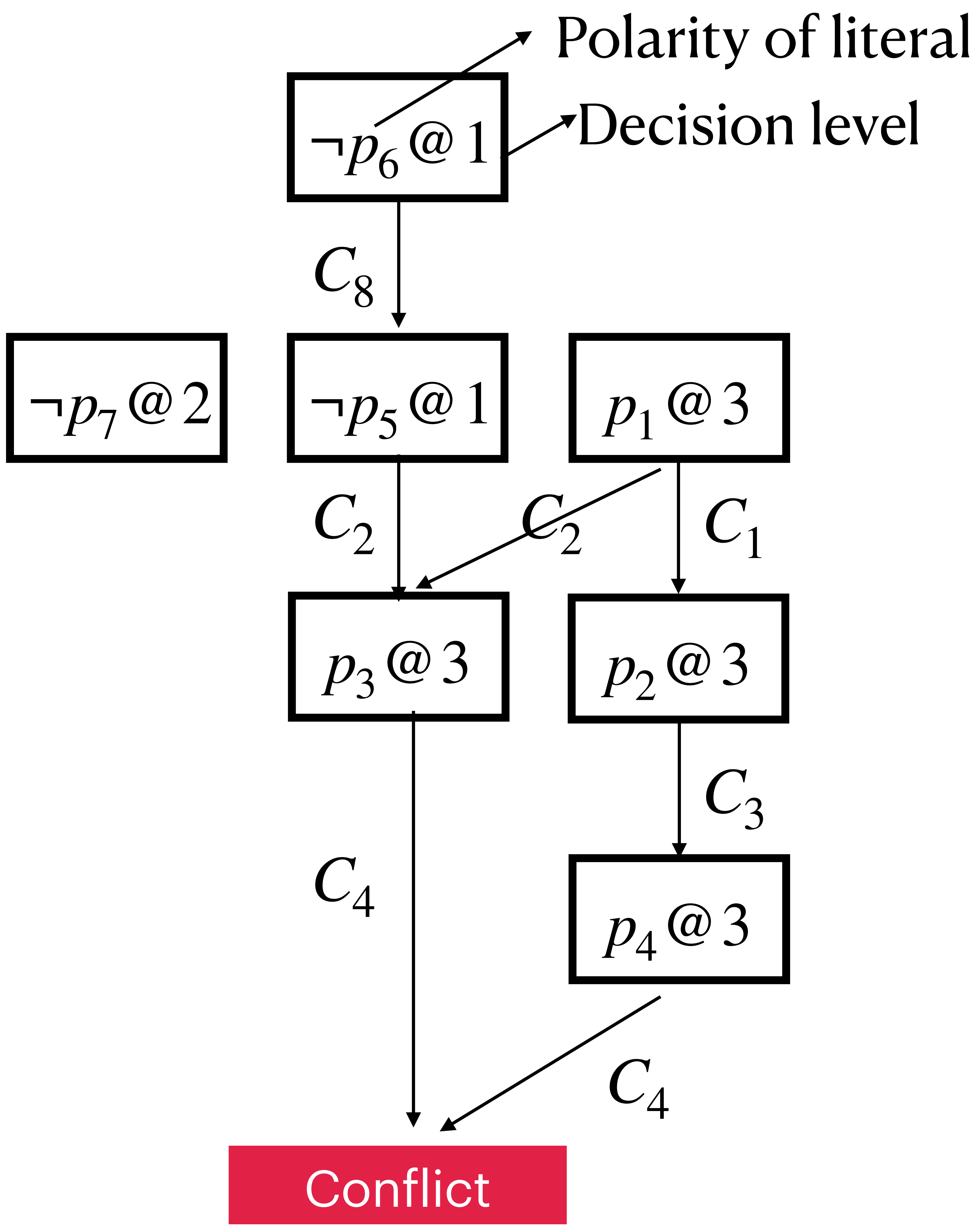
$m(p_6) = 0, m(p_7) = 1, m(p_1) = 1$

$m(p_6) = 0, m(p_1) = 1$

This will never be tried again!



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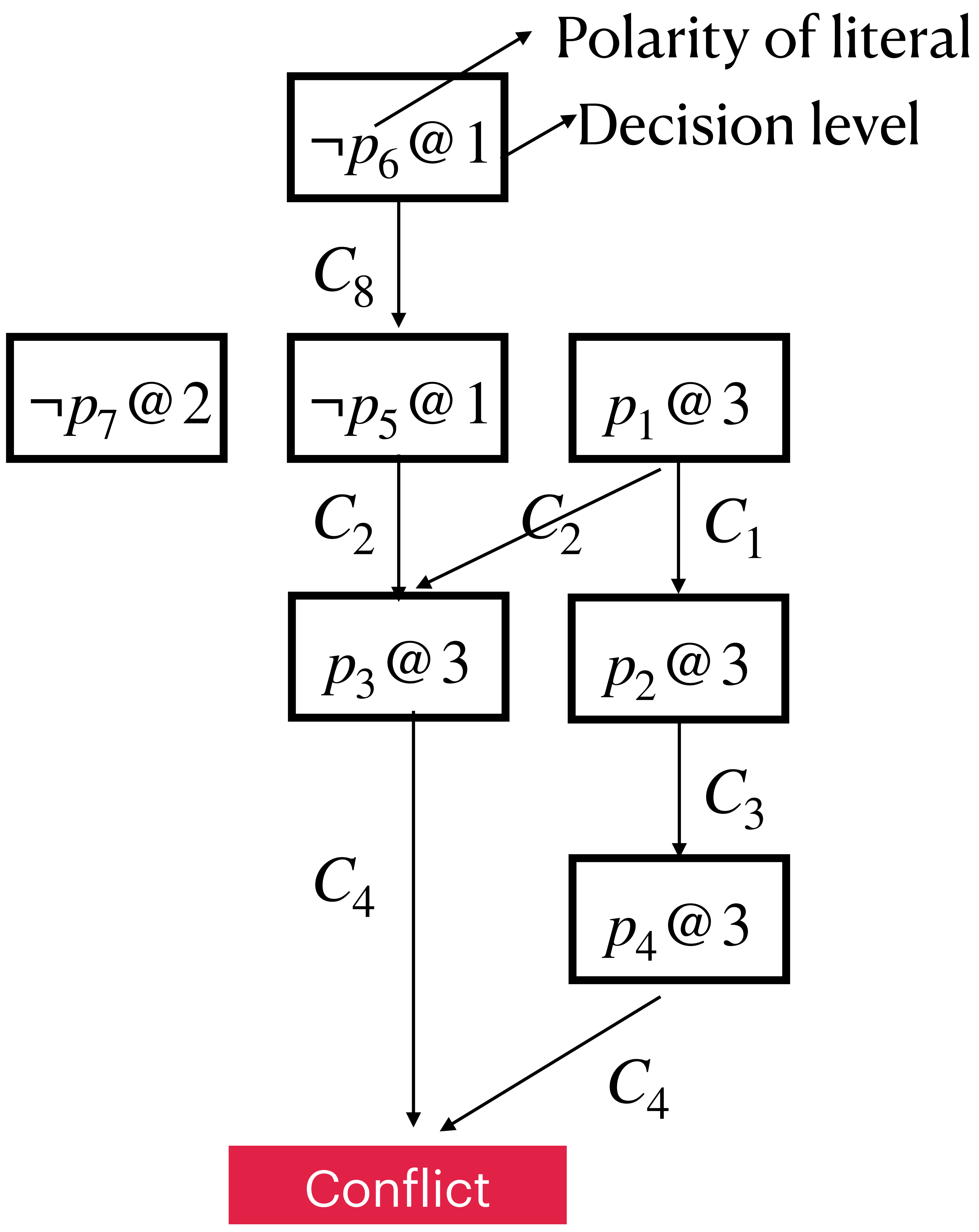
CDCL: Conflict Driven Clause Learning



Implication Graph.

$$C_1 = (\neg p_1 \vee p_2)$$
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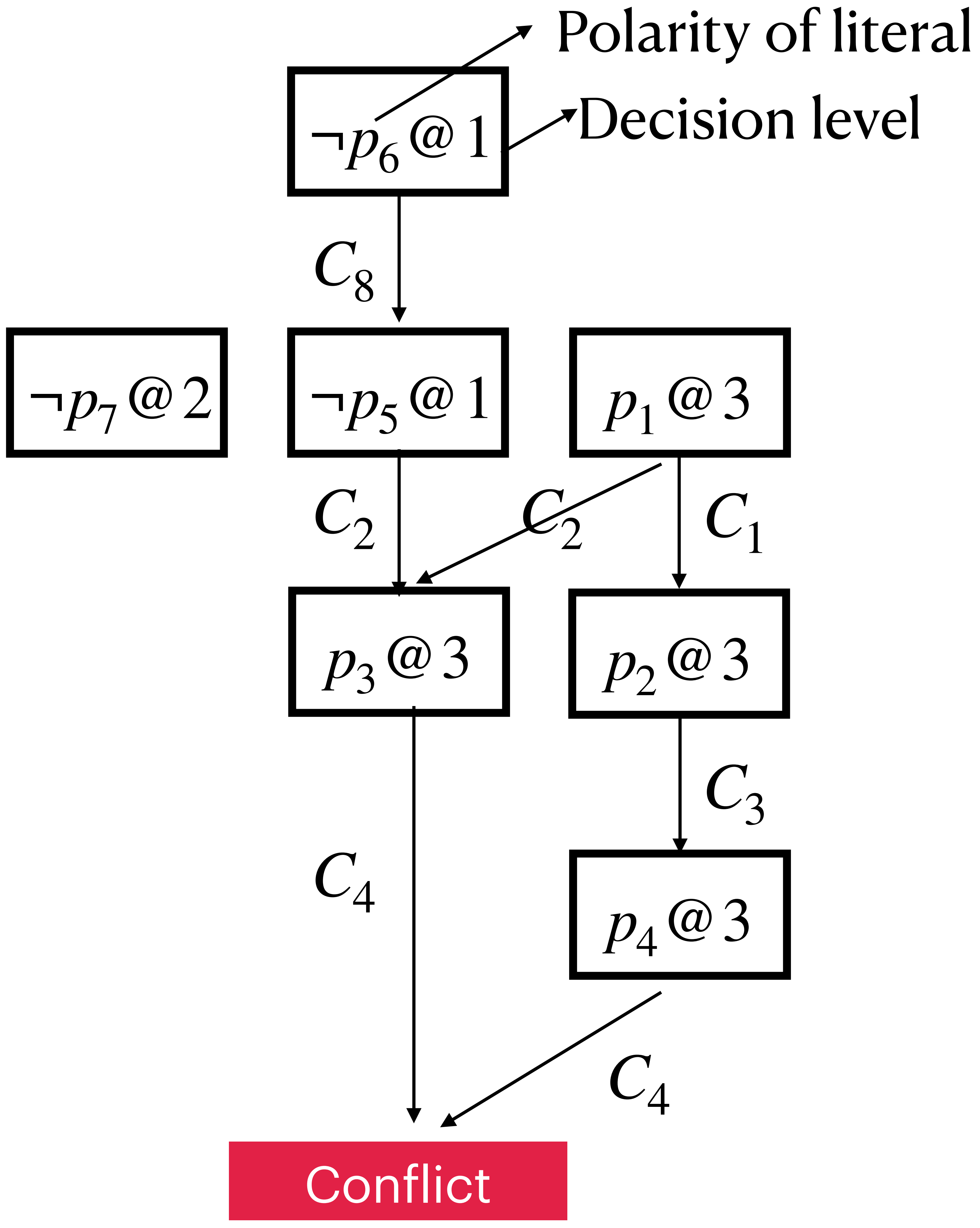

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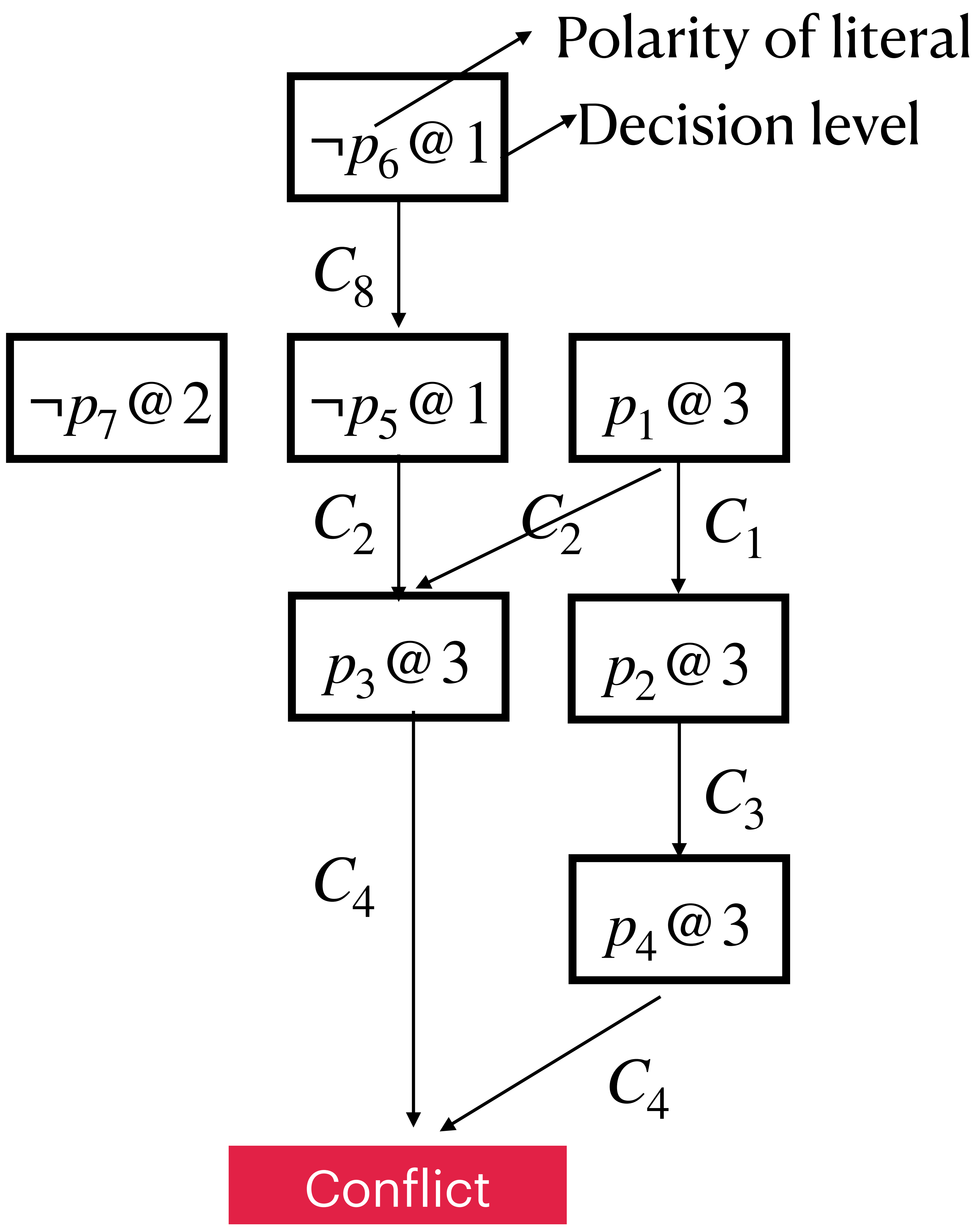
Added a new clause!
Where should we backtrack?



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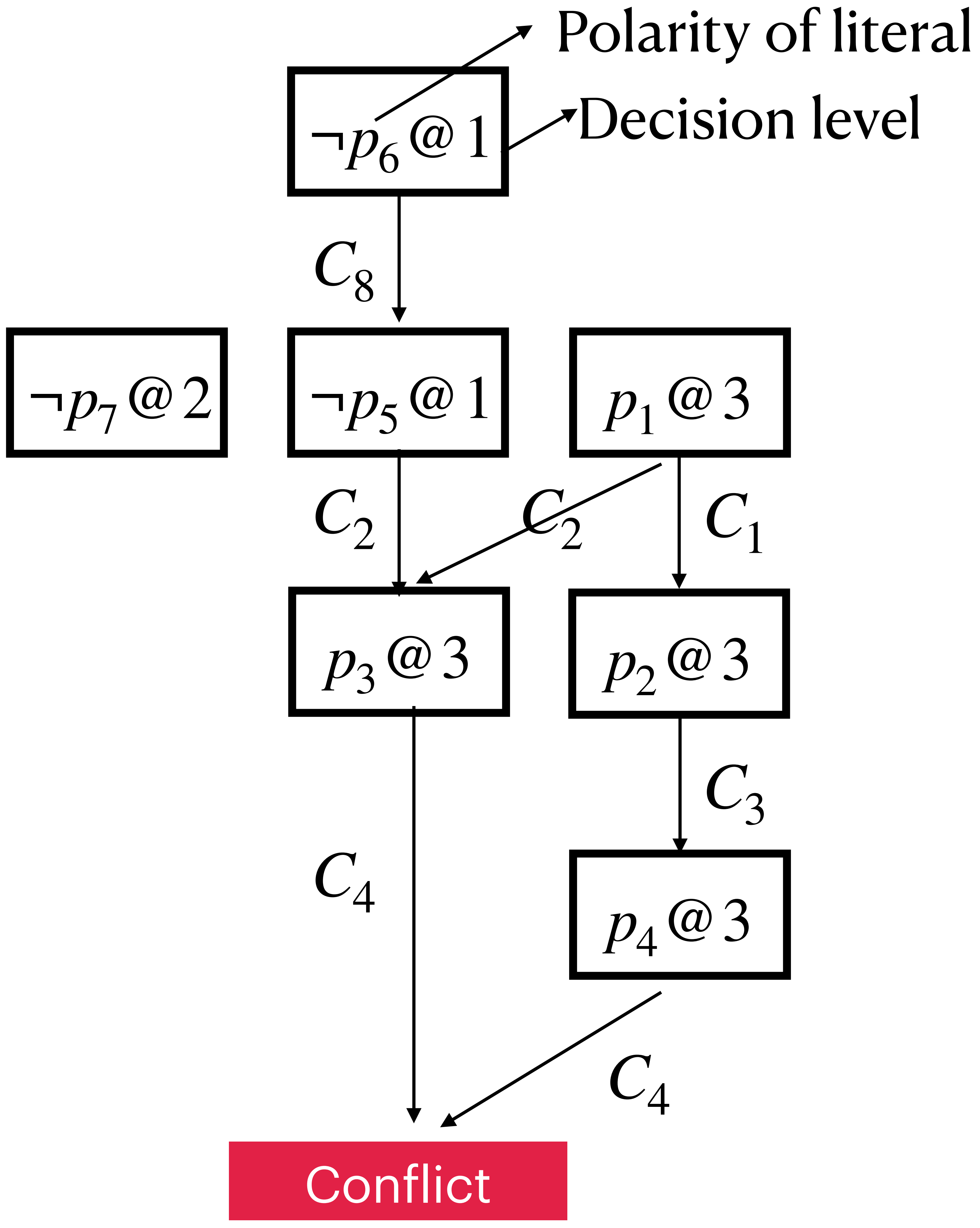
Backtrack to second largest decision in the conflict clause.



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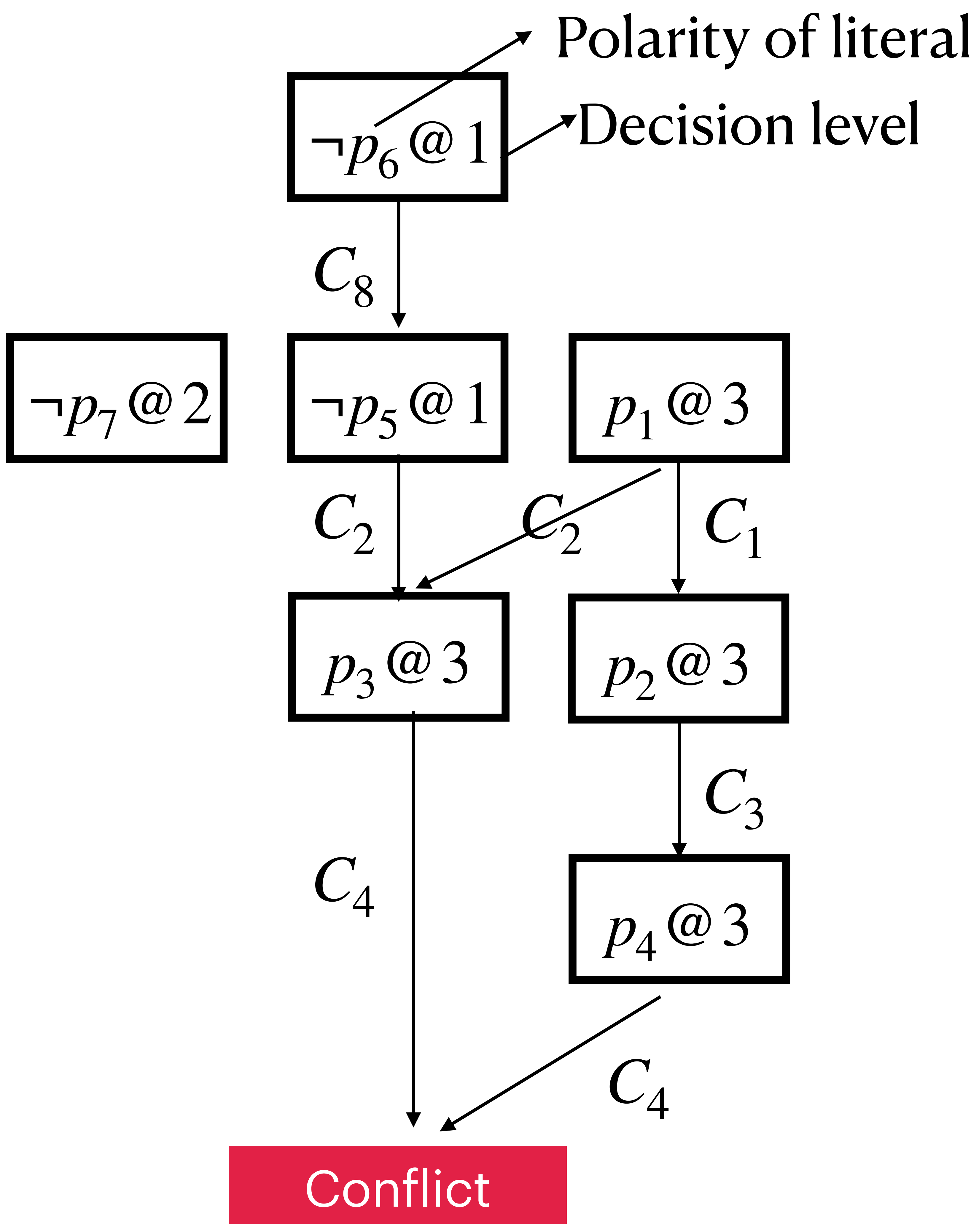
Added a new clause!
Where should we backtrack?

Backtrack to second largest decision in the conflict clause.

Here we should backtrack to decision level 1.



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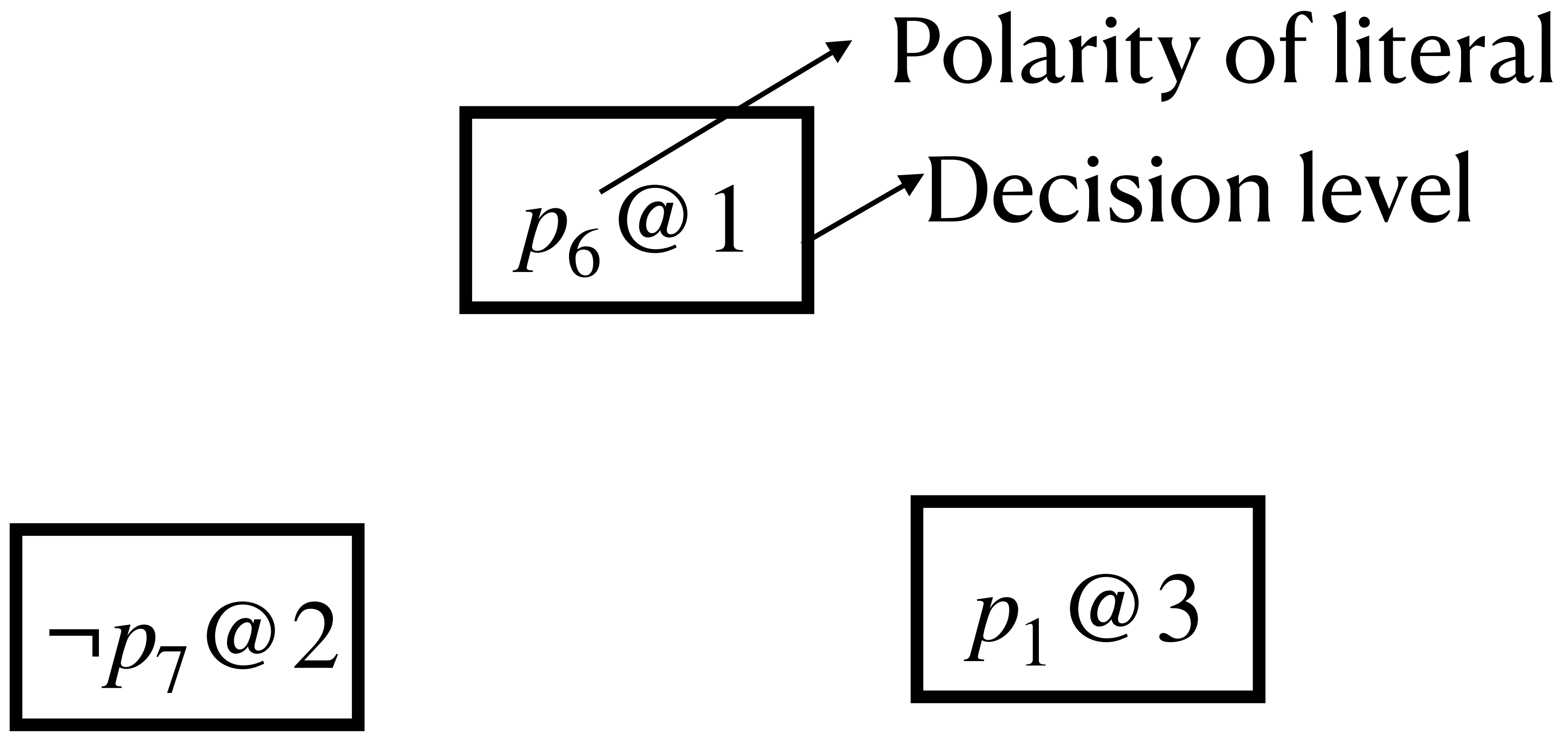

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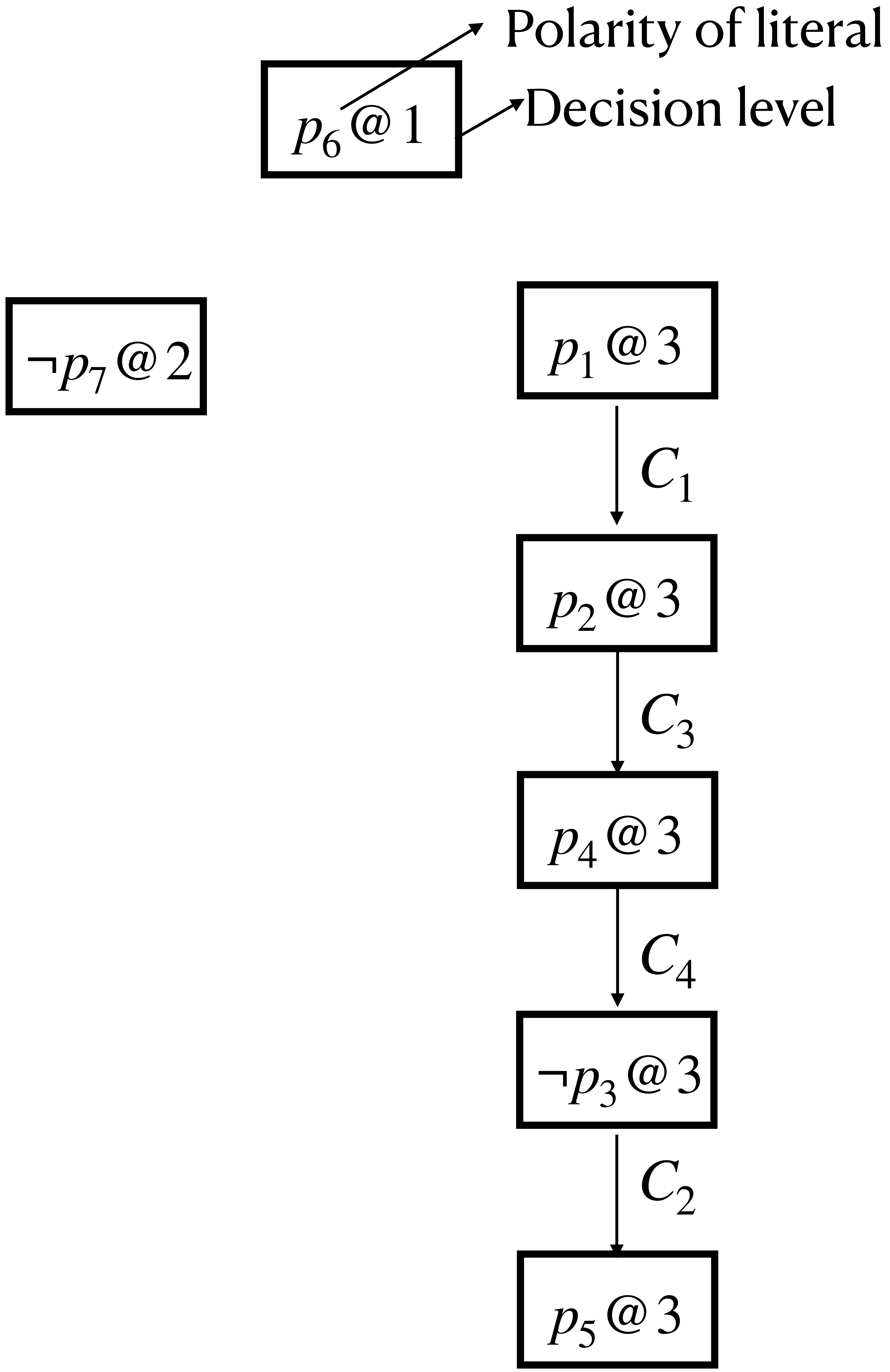

Implication Graph.

CDCL: Conflict Driven Clause Learning

1. UnitPropagation(m, F): applies unit propagation and extends m.

2. Decide(m, F): choose an unassigned variable in m and assign it a Boolean value.

3. ClauseLearning(m, F): returns a conflict clause learned using implication graph, and a decision level upto which the solver needs to backtrack.

Course Webpage

Thanks!