COL:750

Foundations of Automatic Verification

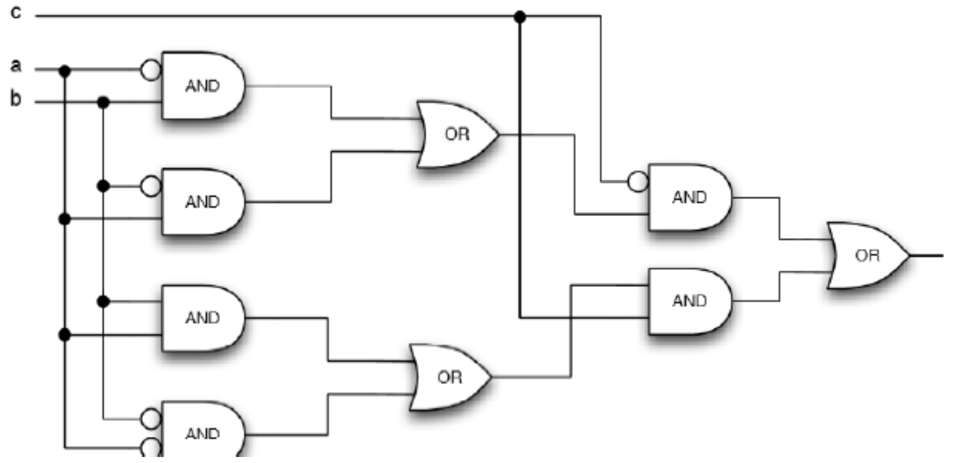
Instructor: Priyanka Golia

Course Webpage

https://priyanka-golia.github.io/teaching/COL-750/index.html

Formal Verification

```
PC1 (char[] SP, char[] UI) {
        for (int i=0; i<UI.length(); i++) {
                if(SP[i] != UI[i]) return No;
        }
        return Yes;
}
```



System

$\models$

Satisfies

Properties

$$S(I,O) \models P(I,O)$$

Is the always the case that S satisfies Property P?

How often S satisfies P?

Why S doesn’t satisfy P?

Why S doesn't satisfy P?

Computing UNSAT core of a formula

UNSAT Core: Given an unsatisfiable Boolean formula F in CNF, a subset of its clauses whose conjunction is also unsatisfiable is called an UNSAT core of F.

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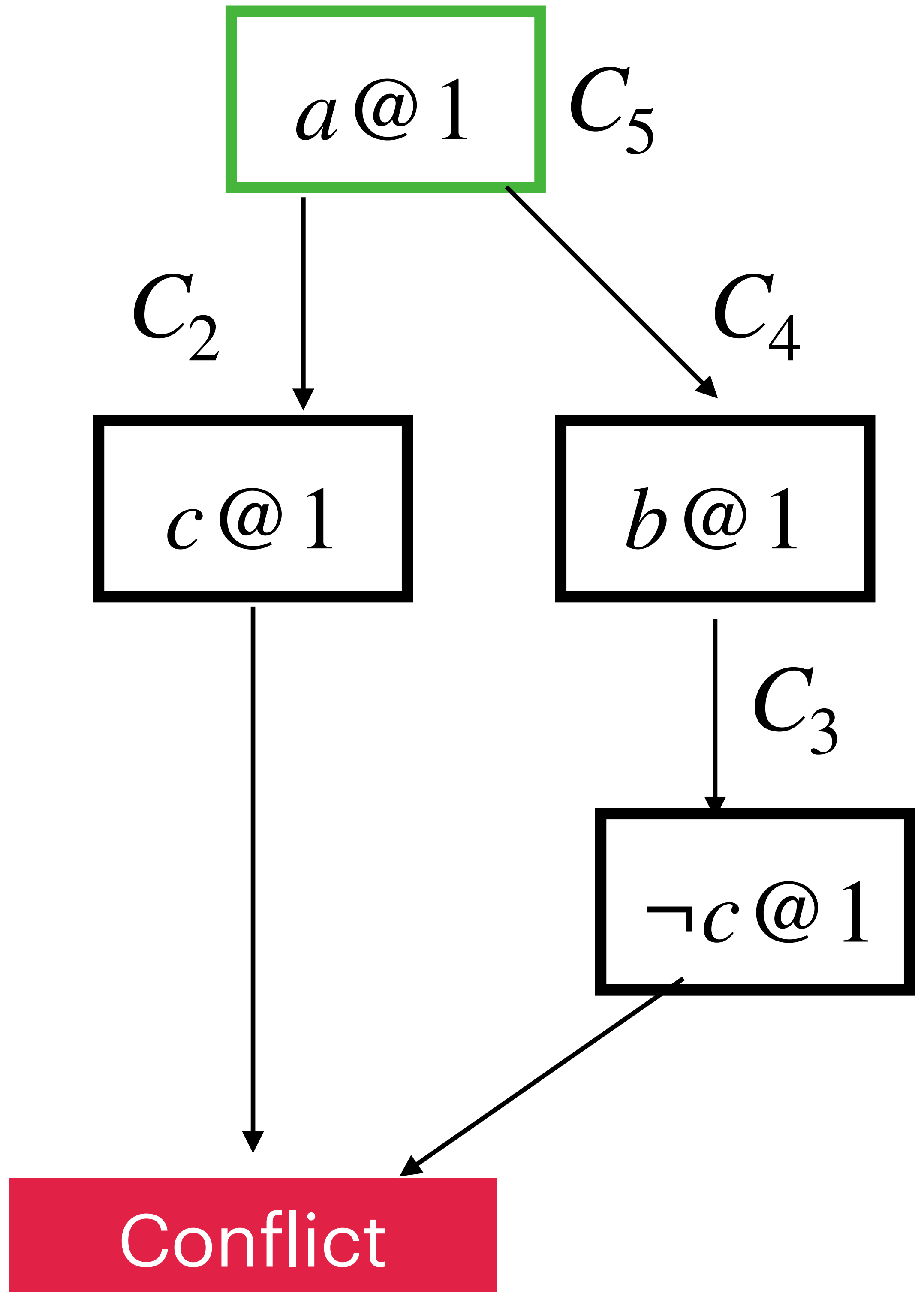
$$F = (a \vee b) \wedge (\neg a \vee c) \wedge (\neg b \vee \neg c) \wedge (\neg a \vee b) \wedge (a)$$

$$\text{UNSAT Core } = \{C_2, C_3, C_4, C_5\}$$



Computing UNSAT core of a formula

UNSAT Core: Given an unsatisfiable Boolean formula F in CNF, a subset of its clauses whose conjunction is also unsatisfiable is called an UNSAT core of F.

$c_1 = a \vee \neg c$ $c_3 = \neg b \vee c$ $c_5 = b \vee c$

$c_2 = b$ $c_4 = \neg b \vee \neg c$ $c_6 = \neg a \vee b \vee \neg c$

$F = c_1 \wedge c_2 \wedge c_3 \wedge c_4 \wedge c_5 \wedge c_6$ How many different unsat cores for F?

$UC_1 = \{c_1, c_2, c_3, c_4, c_5, c_6\}$ $UC_4 = \{c_1, c_2, c_3, c_4, c_6\}$ $UC_7 = \{c_1, c_2, c_3, c_4\}$

$UC_2 = \{c_2, c_3, c_4, c_5, c_6\}$ $UC_5 = \{c_2, c_3, c_4, c_5\}$ $UC_8 = \{c_2, c_3, c_4\}$

$UC_3 = \{c_1, c_2, c_3, c_4, c_5\}$ $UC_6 = \{c_2, c_3, c_4, c_6\}$ $UC_9 = \{c_1, c_3, c_4, c_5, c_6\}$

UNSAT Core
Minimal Unsatisfiable Set.

Consider a subset $M \subseteq C$, where C is a set of all clauses of Formula F

Minimal Unsatisfiable Set (MUS): M is a MUS of F if and only if M is unsatisfiable, **and** all proper subsets of M are satisfiable.

$$F = a \wedge \neg a \wedge b \wedge (\neg a \vee \neg b) \qquad M_1 = \{a, \neg a\} \qquad M_2 = \{a, b, (\neg a \vee \neg b)\}$$

A MUS is an unsatisfiable set that can't be reduced without causing it to become satisfiable.

UNSAT Core

Minimal Unsatisfiable Subset.

$c_1 = a \vee \neg c$ $\quad c_3 = \neg b \vee c$ $\quad c_5 = b \vee c$

$c_2 = b$ $\quad c_4 = \neg b \vee \neg c$ $\quad c_6 = \neg a \vee b \vee \neg c$

$$F = c_1 \wedge c_2 \wedge c_3 \wedge c_4 \wedge c_5 \wedge c_6$$

Minimal unsat cores for F?

$UC_1 = \{c_1, c_2, c_3, c_4, c_5, c_6\}$ $\quad UC_4 = \{c_1, c_2, c_3, c_4, c_6\}$ $\quad UC_7 = \{c_1, c_2, c_3, c_4\}$

$UC_2 = \{c_2, c_3, c_4, c_5, c_6\}$ $\quad UC_5 = \{c_2, c_3, c_4, c_5\}$

$UC_3 = \{c_1, c_2, c_3, c_4, c_5\}$ $\quad UC_6 = \{c_2, c_3, c_4, c_6\}$

$UC_8 = \{c_2, c_3, c_4\}$

$UC_9 = \{c_1, c_3, c_4, c_5, c_6\}$

UNSAT Core Minimal Correction Set.

Consider a subset $M' \subseteq C$, where C is a set of all clauses of Formula F

Minimal Correction Set (MCS): M' is a MCS of F if and only if $C \backslash M'$ is satisfiable, **and** $\forall m \in M', C \backslash \{M' \backslash m\}$ is unsatisfiable.

$F = a \wedge \neg a \wedge b \wedge (\neg a \vee \neg b)$ $\quad M_1' = \{a\}$ $\quad M_2' = \{\neg a, b\}$ $\quad M_3' = \{\neg a, \neg a \vee \neg b\}$

An MCS is a minimal set of clauses whose removal from a formula F makes F satisfiable.

UNSAT Core Minimal Correction Set.

$c_1 = a \vee \neg c$ $c_3 = \neg b \vee c$ $c_5 = b \vee c$

$c_2 = b$ $c_4 = \neg b \vee \neg c$ $c_6 = \neg a \vee b \vee \neg c$

$F = c_1 \wedge c_2 \wedge c_3 \wedge c_4 \wedge c_5 \wedge c_6$ Minimal correction cores for F?

MCS_1 $\{c_3\}$

MCS_2 $\{c_4\}$

How are MUSes and MCSes related?

$F = a \wedge \neg a \wedge b \wedge (\neg a \vee \neg b)$

MUSes $M_1 = \{a, \neg a\}$ $M_2 = \{a, b, (\neg a \vee \neg b)\}$

MCSes $M'_1 = \{a\}$ $M'_2 = \{\neg a, b\}$ $M'_3 = \{\neg a, \neg a \vee \neg b\}$

Hitting Set

A hitting set H of a collection of sets S is a set that "hits" every set in S, that is, $\forall s \in S, H \cap s \neq \emptyset$

H has non empty intersection with every set s of S.

$$S = \{\{a, b\}, \{a, c\}, \{c, d\}\}$$

$$H_1 = \{a, c\} \qquad H_2 = \{a, b, c\} \qquad H_3 = \{a, c, d\} \qquad H_4 = \{a, d\}$$

A minimal hitting set is a hitting set such that no strict subset of it is also a hitting set.

$$H_1 = \{a, c\} \qquad H_4 = \{a, d\}$$

MUSes and MCSes

Every MCS is a minimal hitting set of the set of MUSes

Every MUS is a minimal hitting set of the set of MCSes.

$$F = a \wedge \neg a \wedge b \wedge (\neg a \vee \neg b)$$

MUSes $M_1 = \{a, \neg a\}$ $M_2 = \{a, b, (\neg a \vee \neg b)\}$

MCSes $M'_1 = \{a\}$ $M'_2 = \{\neg a, b\}$ $M'_3 = \{\neg a, \neg a \vee \neg b\}$

Critical Clauses

Consider a subset $C' \subseteq C$, where C is a set of all clauses.

C' is a said to be Critical for formula F if :

1. C' must be contained in every MUS of F.
2. C' is an MCS of F.

Removal of C' from F, causes F to become satisfiable

$F = a \wedge \neg a \wedge b \wedge (\neg a \vee \neg b)$

$\{a\}$ is a critical clause.

MUSes $M_1 = \{a, \neg a\}$ $M_2 = \{a, b, (\neg a \vee \neg b)\}$

MCSes $M_1' = \{a\}$ $M_2' = \{\neg a, b\}$ $M_3' = \{\neg a, \neg a \vee \neg b\}$

Can we come up with an algorithm to find MUS?

Computing MUS

Key Observation: each clause is a critical clause in MUS

Find an UNSAT Core UC.

For each clauses $c \in UC$

If c is NOT a critical clause in UC

$$UC \leftarrow UC \backslash \{c\}$$

Return UC

How do we check if clause is a critical clause or not ?

Computing MUS

Key Observation: each clause is a critical clause in MUS

Find an UNSAT Core UC.

For each clauses $c \in UC$

If NOT CheckSAT($F \backslash \{c\}$)

$$UC \leftarrow UC \backslash \{c\}$$

Return UC

$$F = a \wedge \neg a \wedge b \wedge (\neg a \vee \neg b)$$

Computing MUS

Key Observation: each clause is a critical clause in MUS

Find an UNSAT Core UC.

UnknownClauses $\leftarrow$ $Clauses(UC)$

CriticalClauses $\leftarrow \emptyset$

While ($UnknownClauses \neq \emptyset$) {

Choose a clauses c in UnknownClauses

UnknownClauses $\leftarrow$ UnknownClauses $\setminus$ c

$(SAT?, \sigma, UC') \leftarrow$ CheckSAT($UnknownClauses \cup CriticalClauses$)

If SAT:

CriticalClauses $\leftarrow$ CriticalClauses $\cup\, c$

Else:

UnknownClauses $\leftarrow$ UnknownClauses $\cap$ Clauses(UC')

}

Return CriticalClauses

Adding a single clause
at a time to Critical Clauses!!
Can we do better?

Critical Clauses

$$(SAT?, \sigma, UC') \leftarrow \text{CheckSAT}(UnknownClauses \cup CriticalClauses \cup \neg c)$$

$$\sigma \nvDash c$$

Let there be a another satisfying assignment σ' such that $\sigma' = \sigma_{\downarrow(\neg v)}$ where $v \in Vars(c)$

$\sigma' \vDash c \quad \sigma' \nvDash UC$ *UC is UNSAT*

$\exists C' \in UC \backslash c, s.t., \sigma' \nvDash C'$ There has to be at least a clause c' in $UC \backslash c$, such that, $\sigma' \nvDash c'$

Clauses in C' are also critical clauses.

Critical Clauses

Recursive Model Rotation (RMR)

$\text{UC} = (\neg a \vee \neg b) \wedge (b \vee \neg c) \wedge (a \vee b) \wedge (a \vee \neg b) \wedge (b \vee c)$

Check if $(\neg a \vee \neg b)$ is a critical clause or not?

$\text{CheckSAT}(UC \backslash (\neg a \vee \neg b) \wedge (a \wedge b))$

CheckSAT($UnknownClauses \cup CriticalClauses \cup \neg c$)
UnknownClauses = $UC \backslash c$, Critical Clauses = $\emptyset$

$\sigma = \langle a = 1, b = 1, c = 1 >$

$\sigma' = \sigma_{\downarrow \neg v}, v \in Vars(c) \ is \ \langle a = 0, b = 0 \rangle$

$\sigma' \vDash c$ $\sigma' \nvDash UC$

$\sigma' \nvDash (a \vee b)$ This is also a critical clause.

Computing MUS

Key Observation: each clause is a critical clause in MUS

Find an UNSAT Core UC.

UnknownClauses, CriticalClauses $\leftarrow$ $Clauses(UC), \emptyset$

While ($UnknownClauses \neq \emptyset$) {

Choose a clauses c in UnknownClauses

UnknownClauses $\leftarrow$ UnknownClauses $\setminus$ c

$(SAT?, \sigma, UC') \leftarrow$ CheckSAT($UnknownClasues \cup CriticalClauses \cup \neg c$)

If SAT:

CriticalClauses $\leftarrow$ CriticalClauses $\cup$ c

MoreCriticalClauses ← RMR($\sigma, c, Unknowclauses, CriticalClasues$)

CriticalClauses ← CriticalClauses ∪ MoreCriticalClauses

UnknownClauses ← UnknownClauses \ MoreCriticalClauses

Else:

UnknownClauses $\leftarrow$ UnknownClauses $\cap$ Clauses(UC')

}

Return CriticalClauses

Computing MUS

$$F = a \wedge \neg a \wedge b \wedge (\neg a \vee \neg b)$$

UnknownClauses = $\{a, \neg a, b, \neg a \vee \neg b\}$ Critical Clauses $\{\varnothing\}$

Computing MUS

$$F = a \wedge \neg a \wedge b \wedge (\neg a \vee \neg b)$$

UnknownClauses = $\{a, \neg a, b, \neg a \vee \neg b\}$ Critical Clauses $\{\emptyset\}$

1. Choose $\{a\}$, check if it is critical

$$\text{CheckSAT}\ ((\neg a \wedge b \wedge (\neg a \vee \neg b)) \wedge \neg a)$$

It is SAT. $\sigma = \langle a = 0, b = 1 \rangle$

2. Look for other critical clauses.

$$\sigma' = \sigma_{\downarrow(\neg(a=0))} = \langle a = 1 \rangle$$

$\sigma' \not\models \neg a$ This will also be added to critical clauses.

UnknownClauses = $\{b, \neg a \vee \neg b\}$ Critical Clauses $\{a, \neg a\}$

Computing MUS

$$F = a \wedge \neg a \wedge b \wedge (\neg a \vee \neg b)$$

UnknownClauses = $\{b, \neg a \vee \neg b\}$ Critical Clauses $\{a, \neg a\}$

1. Choose $\{b\}$, check if it is critical

CheckSAT $(((a \wedge \neg a \wedge (\neg a \vee \neg b)) \wedge \neg b))$ It is UNSAT.

UnknownClauses = $\{\neg a \vee \neg b\}$ Critical Clauses $\{a, \neg a\}$

1. Choose $\{\neg a \vee \neg b\}$, check if it is critical

CheckSAT $((a \wedge \neg a) \wedge \neg(\neg a \vee \neg b))$ It is UNSAT.

UnknownClauses = $\varnothing$ Critical Clauses $\{a, \neg a\}$ MUS: $\{a, \neg a\}$