COL:750

Foundations of Automatic Verification

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Course Webpage

https://priyanka-golia.github.io/teaching/COL-750/index.html

If formula is SATisfiable, gives an satisfying assignment

Boolean /propositional formulas ——> SAT Solvers

UNSAT

Equisatisfiable Formulas (modified)

Boolean (propositional) formulas F and G are equisatisfiable if the following holds:

$Vars(G) \subseteq Vars(F)$

- Every satisfying assignment of G can be extended to the satisfying assignment of F.
    - For every $\tau \vDash G$, there is a τ' such that τ' extends τ to $Vars(F/G)$, and $\tau' \vDash F$
- Every satisfying assignment of F can be projected on $Vars(G)$ to get the satisfying assignment of G.
    - For every $\tau' \vDash F$, there is a τ such that $\tau = \tau'_{\downarrow Vars(G)}$ and $\tau \vDash G$

Equisatisfiable Formulas (modified)

$$F = (p \vee \alpha) \wedge (\neg p \vee \beta) \quad \text{and} \quad G = (\alpha \vee \beta)$$

$$Models(F) := \{(p \mapsto 1, \alpha \mapsto 0, \beta \mapsto 1), (p \mapsto 1, \alpha \mapsto 1, \beta \mapsto 1), (p \mapsto 0, \alpha \mapsto 1, \beta \mapsto 0), (p \mapsto 0, \alpha \mapsto 1, \beta \mapsto 1)\}$$

$$Models(F)_{\downarrow Vars(G)} := \{(\alpha \mapsto 0, \beta \mapsto 1), (\alpha \mapsto 1, \beta \mapsto 1), (\alpha \mapsto 1, \beta \mapsto 0)\}$$

$$Models(F)_{\downarrow Vars(G)} := Models(G)$$

For every $\tau \vDash G$, there is a τ' such that τ' extends τ to $Vars(F/G)$, and $\tau' \vDash F$

For every $\tau' \vDash F$, there is a τ such that $\tau = \tau'_{\downarrow Vars(G)}$ and $\tau \vDash G$

Equisatisfiable Formulas (modified)

$G = p \vee (q \wedge r)$

$F = (p \vee t) \wedge (t \leftrightarrow q \wedge r)$

$F' = (p \vee t) \wedge (t \rightarrow q \wedge r)$

Is F and G equisatisfiable?

Is F’ and G equisatisfiable?

Exercise:

$G = (x_1 \wedge y_1) \vee (x_2 \wedge y_2)$

F $= (t_1 \vee t_2) \wedge (t_1 \leftrightarrow (x_1 \wedge y_1) \wedge (t_2 \leftrightarrow (x_2 \wedge y_2)$

F’ $= (t_1 \vee t_2) \wedge (t_1 \rightarrow (x_1 \wedge y_1) \wedge (t_2 \rightarrow (x_2 \wedge y_2)$

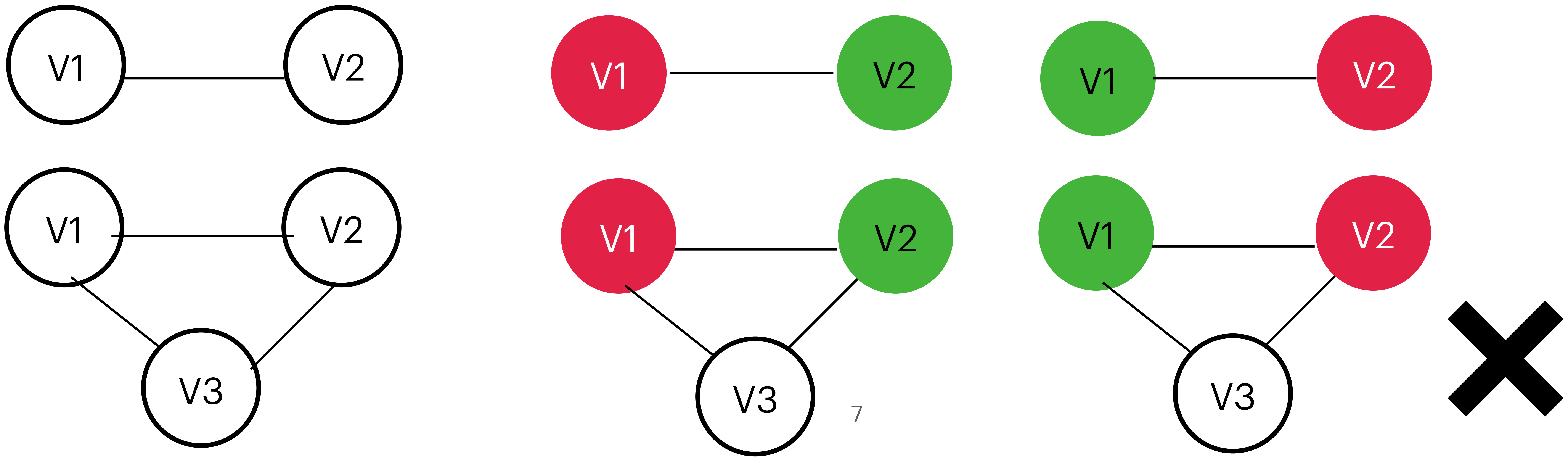
Is F and G equisatisfiable?

Is F’ and G equisatisfiable?

Constraint Encoding

Encoding of Graph Coloring to SAT

- Proper coloring: An assignment of colors to the vertices of a graph such that no two adjacent vertices have same color.
- K-color: A proper coloring involving a total of K colors.
- Is the following graphs 2-colorable?

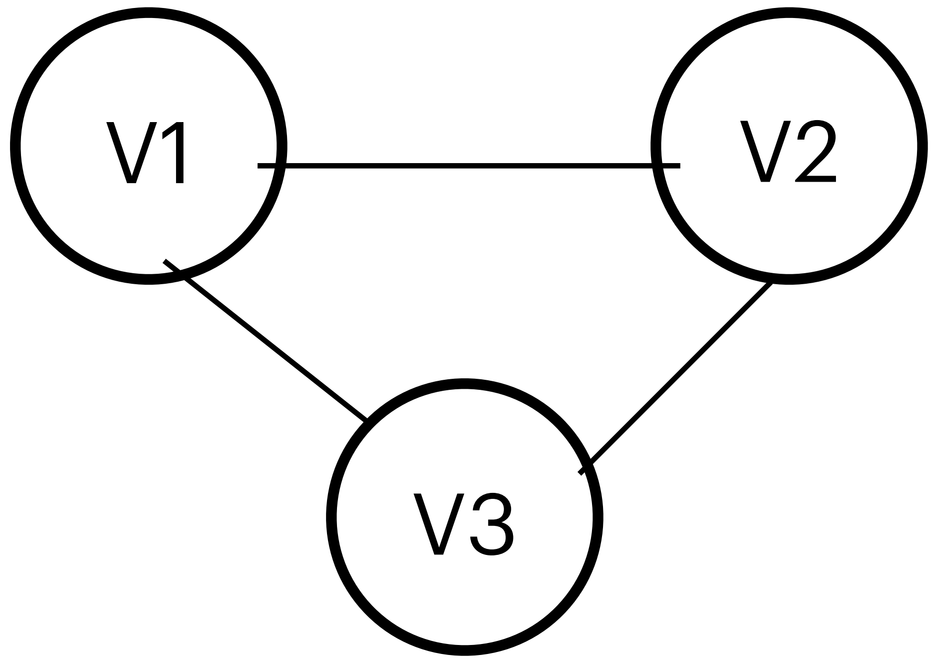


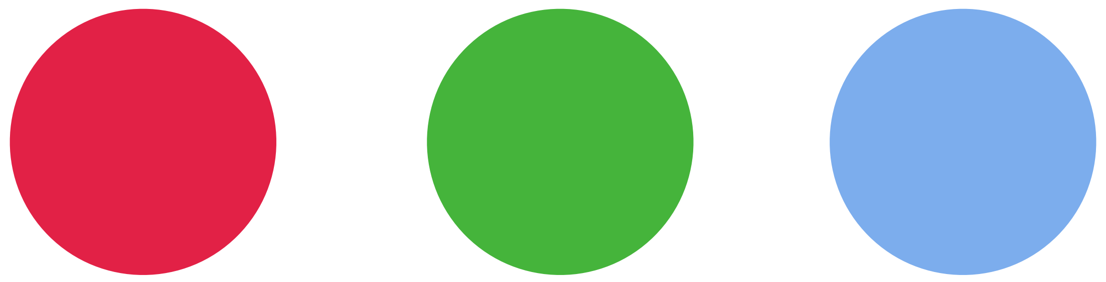
Encoding of Graph Coloring to SAT

Given a graph G(V,E) with V as a set of vertices and E as a set of edges, and an integer K (representing the number of colors), can we encode the proper graph coloring into a CNF formula such that the formula is satisfiable (SAT) if and only if the graph is K-colorable.

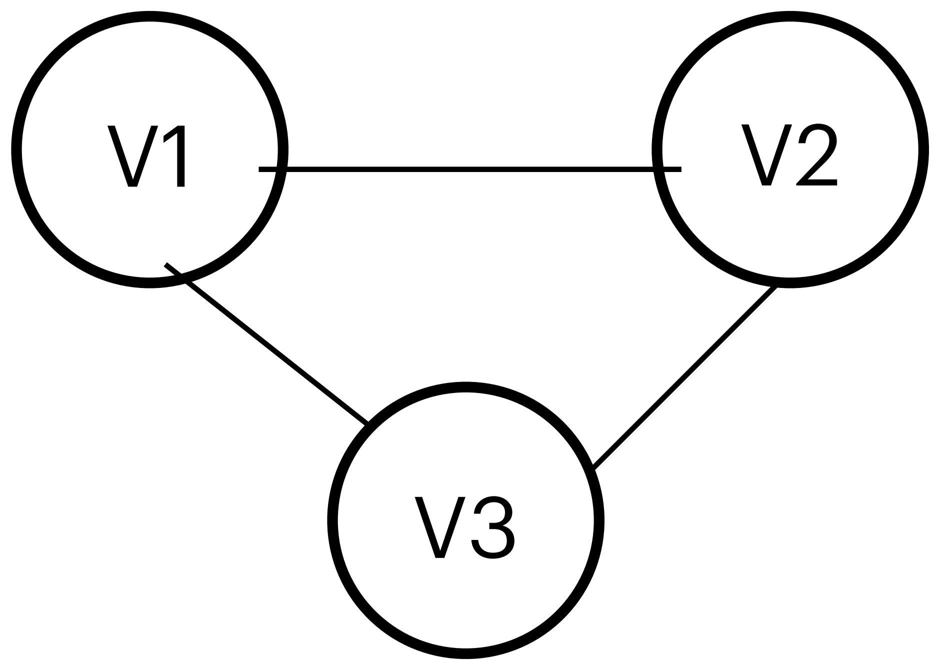
We want to encode that:

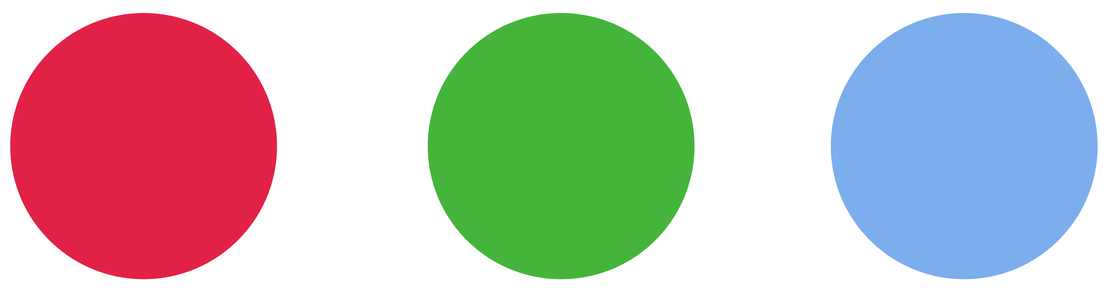
- No two adjacent vertices share the same color.
- Each vertex has exactly one color.





Step 1: Propositional Variables

- Use propositional variables $v_{i,g}$, where $i \in \{1,2,3\}, g \in \{R, G, B\}$
- $v_{i,g}$ is True, if and only if, vertex i is assigned g color.
- 



$v_{1,G}, v_{1,R}, v_{1,B}$ $v_{2,G}, v_{2,R}, v_{2,B}$ $v_{3,G}, v_{3,R}, v_{3,B}$

Step 2: Encoding Constraints

- Each vertex must have exactly one color.
    - Each vertex must have at least one color, and each vertex must have at most one color

How are we going to encode, each vertex must have at least one color:

For vertex $V_1 : v_{1,G} \vee v_{1,R} \vee v_{1,B}$ $\quad V_2 : v_{2,G} \vee v_{2,R} \vee v_{2,B}$ $\quad V_3 : v_{3,G} \vee v_{3,R} \vee v_{3,B}$

How are we going to encode, each vertex must have at most one color:

$V_1 : (\neg v_{1,G} \vee \neg v_{1,R}) \wedge$ $\quad V_2 : (\neg v_{2,G} \vee \neg v_{2,R}) \wedge$ $\quad V_3 : (\neg v_{3,G} \vee \neg v_{3,R}) \wedge$

$(\neg v_{1,G} \vee \neg v_{1,B}) \wedge$ $\quad (\neg v_{2,G} \vee \neg v_{2,B}) \wedge$ $\quad (\neg v_{3,G} \vee \neg v_{3,B}) \wedge$

$(\neg v_{1,R} \vee \neg v_{1,B}) \wedge$ $\quad (\neg v_{2,R} \vee \neg v_{2,B}) \wedge$ $\quad (\neg v_{3,R} \vee \neg v_{3,B}) \wedge$

Step 2: Encoding Constraints

- No two adjacent vertex have the same color.

For V_1 and V_2:

$(\neg v_{1,R} \vee \neg v_{2,R}) \wedge$

$(\neg v_{1,G} \vee \neg v_{2,G}) \wedge$

$(\neg v_{1,B} \vee \neg v_{2,B}) \wedge$

For V_1 and V_3:

$(\neg v_{1,R} \vee \neg v_{3,R}) \wedge$

$(\neg v_{1,G} \vee \neg v_{3,G}) \wedge$

$(\neg v_{1,B} \vee \neg v_{3,B}) \wedge$

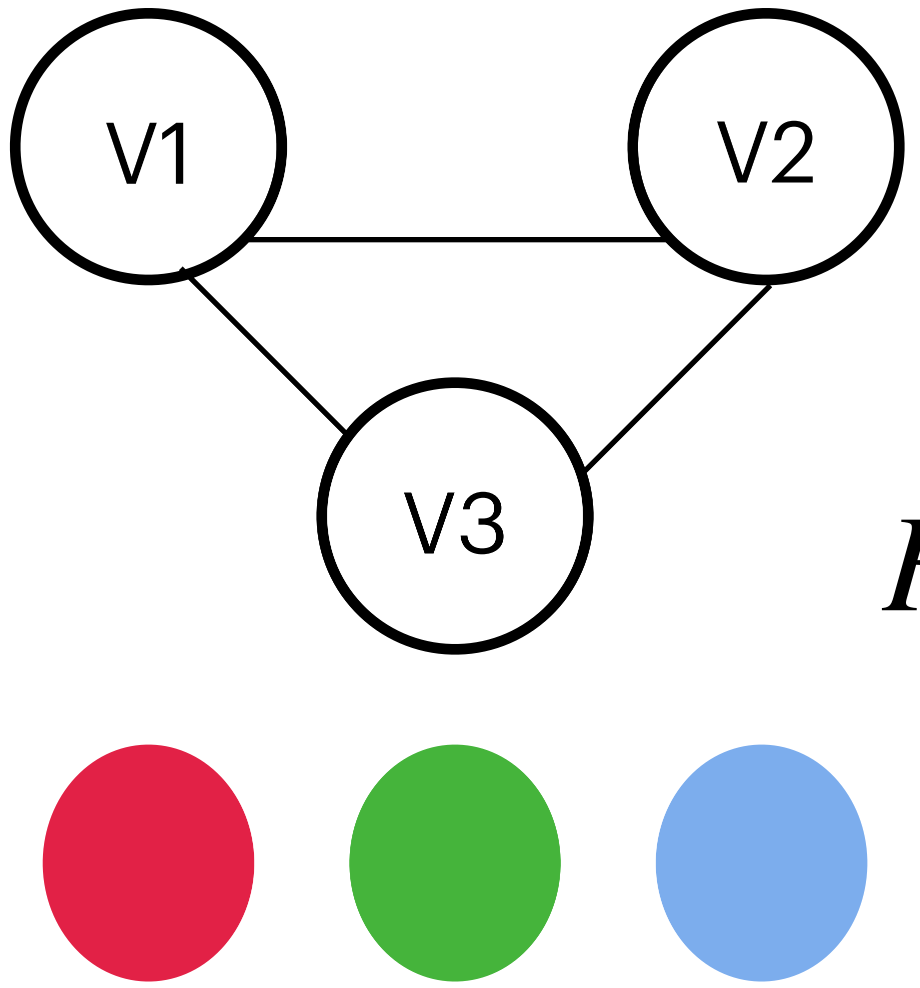
For V_2 and V_3:

$(\neg v_{2,R} \vee \neg v_{3,R}) \wedge$

$(\neg v_{2,G} \vee \neg v_{3,G}) \wedge$

$(\neg v_{2,B} \vee \neg v_{3,B})$

Proper Coloring to SAT



$$
\begin{aligned}
F_{CNF} = \; & (v_{1,G} \lor v_{1,R} \lor v_{1,B}) \land (v_{2,G} \lor v_{2,R} \lor v_{2,B}) \land (v_{3,G} \lor v_{3,R} \lor v_{3,B}) \land \\
& (\neg v_{1,G} \lor \neg v_{1,R}) \land (\neg v_{1,G} \lor \neg v_{1,B}) \land (\neg v_{1,R} \lor \neg v_{1,B}) \land \\
& (\neg v_{2,G} \lor \neg v_{2,R}) \land (\neg v_{2,G} \lor \neg v_{2,B}) \land (\neg v_{2,R} \lor \neg v_{2,B}) \land \\
& (\neg v_{3,G} \lor \neg v_{3,R}) \land (\neg v_{3,R} \lor \neg v_{3,B}) \land (\neg v_{3,R} \lor \neg v_{3,B}) \land \\
& (\neg v_{1,R} \lor \neg v_{2,R}) \land (\neg v_{1,G} \lor \neg v_{2,G}) \land (\neg v_{1,B} \lor \neg v_{2,B}) \land \\
& (\neg v_{1,R} \lor \neg v_{3,R}) \land (\neg v_{1,G} \lor \neg v_{3,G}) \land (\neg v_{1,B} \lor \neg v_{3,B}) \land \\
& (\neg v_{2,R} \lor \neg v_{3,R}) \land (\neg v_{2,G} \lor \neg v_{3,G}) \land (\neg v_{2,B} \lor \neg v_{3,B})
\end{aligned}
$$

Encoding of Pigeon Hole Principle to SAT

Theorem: If we place n+1 pigeons in n holes then there is a hole with at least 2 pigeons

Thm is true for any n; can we prove it for a fixed n using SAT solvers?

Exercise:

Encode Pigeon hole principle to a CNF formula for 3 pigeons and 2 holes.
The CNF formula should be SAT if and if 3 pigeons can fit in 2 holes, otherwise formula should be UNSAT.

Boolean /propositional formulas ——> SAT Solvers

If formula is SATisfiable, gives an satisfying assignment

UNSAT

SAT Solvers

Given a formula F, can we determine whether it is satisfiable?

Let F is over X variables, where $X = \{x_1, x_2, \ldots, x_n\}$.

CheckSAT(F){

For τ in 2^n {

If $F(\tau) = 1$ then:

Return SAT, τ

}

Return UNSAT

}

Can we do better ?

We don't know!

Resolution Refutation

List of clauses $C_1, C_2, \ldots, C_t$ is a resolution refutation of formula F_{CNF} if:

1. C_t is empty $\square$
2. $C_K \in F_{CNF}$ or C_k is derived using resolution from C_i and C_j, where $i, j < k$

$Models(F) = \varnothing$
F is UNSAT

$C_i = p \vee \alpha$
$C_j = \neg p \vee \beta$
Then,
$C_k = \alpha \vee \beta$

C_k is derived from C_i, C_j

Resolution Refutation

$$F = \underbrace{(\neg p \vee \neg q \vee r)}_{C_1} \wedge \underbrace{(\neg p \vee q)}_{C_2} \wedge \underbrace{(p)}_{C_3} \wedge \underbrace{(\neg r)}_{C_4}$$

Resolution on C_1, C_3 $\quad \dfrac{(\neg p \vee \neg q \vee r) \wedge (p)}{C_5 : (\neg q \vee r)}$

Resolution on C_2, C_3 $\quad \dfrac{(\neg p \vee q) \wedge (p)}{C_6 : (q)}$

Resolution on C_5, C_4 $\quad \dfrac{(\neg q \vee r) \wedge (\neg r)}{C_7 : (\neg q)}$

Resolution on C_6, C_7 $\quad \dfrac{(q) \wedge (\neg q)}{C_8 : \square}$

List of clauses $C_1, C_2, \ldots, C_8$ is a resolution refutation of F

Resolution Refutation

Thm: A formula F_{CNF} is refutable if and only if F_{CNF} is unsatisfiable

$\rightarrow$ direction is easy to see: if F_{CNF} is refutable then F_{CNF} is unsatisfiable.

HW:

$\leftarrow$ direction: if F_{CNF} is unsatisfiable then F_{CNF} is refutable

Hint: Induction on # of propositional variables.

SAT Solving using Resolution

1. Start with F_{CNF}
2. Perform Resolution until
   1. Empty clause is derived —> return UNSAT
   2. No further resolution is possible —-> return SAT

One of these two cases will occur — resolution is sound and complete.

Bottleneck of Resolution Refutation

Space required to preform Resolutions:

1. At every resolutions step: $\binom{m}{2}$ new clauses are added to the formula, where m is number of clauses.
2. This is done linear many times (O($Vars(F)$) many), hence over growth can be exponential.
3. Resolution is EXPSPACE.

DP algorithm for SAT Solving (Martin Davis - Hilary Putnam 1960)

1. Start with F_{CNF}
2. Pick a literal l that occurs with both polarities in F_{CNF} .
3. For every clause C in F_{CNF} containing l and every clause C' in F_{CNF} containing its negation $\neg l$ perform resolution
    1. $r = (C \setminus \{l\}) \cup (C' \setminus \{\neg l\})$
    2. $F_{CNF} \leftarrow add\_to\_formula(r, F_{CNF})$
4. For every clause C that contains l or $\neg l$ do
    1. $F_{CNF} \leftarrow remove\_from\_formula(C, F_{CNF})$

DP algorithm for SAT Solving (Martin Davis - Hilary Putnam 1960)

1. Start with F_{CNF}
2. Pick a literal l that occurs with both polarities in F_{CNF} .

$F_{CNF} \leftarrow Resolution(C, l, F_{CNF})$

3. For every clause C in F_{CNF} containing l and every clause C' in F_{CNF} containing its negation $\neg l$ perform resolution
   1. $r = (C \backslash \{l\}) \cup (C' \backslash \{\neg l\})$
   2. $F_{CNF} \leftarrow add\_to\_formula(r, F_{CNF})$
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DP algorithm for SAT Solving (Martin Davis - Hilary Putnam 1960)

1. Start with F_{CNF}
2. If F_{CNF} has empty clause then
    1. Return UNSAT
3. If $\nexists l$ that occurs with both polarities in different clauses in F_{CNF}
    1. Return SAT
3. Pick a literal l that occurs with both polarities in F_{CNF} .
    1. $F_{CNF} \leftarrow Resolution(C, l, F_{CNF})$
4. For every clause C that contains l or $\neg l$ do :
    1. $F_{CNF} \leftarrow remove\_from\_formula(C, F_{CNF})$

Is this correct?

How about $(p \vee \neg p)$

DP algorithm for SAT Solving (Martin Davis - Hilary Putnam 1960)

1. Start with F_{CNF}
2. For every clause C in F_{CNF} that contains both l and $\neg l$ do:
    1. $F_{CNF} \leftarrow remove\_from\_formula(C, F_{CNF})$
3. If F_{CNF} is empty
    1. Return SAT
4. If F_{CNF} has empty clause then
    1. Return UNSAT
5. If $\nexists l$ that occurs with both polarities in different clauses in F_{CNF}
    1. Return SAT
6. Pick a literal l that occurs with both polarities in F_{CNF} .
    1. $F_{CNF} \leftarrow Resolution(C, l, F_{CNF})$
7. For every clause C that contains l or $\neg l$ do :
    1. $F_{CNF} \leftarrow remove\_from\_formula(C, F_{CNF})$

DP algorithm for SAT Solving (Martin Davis - Hilary Putnam 1960)

1. Start with F_{CNF}
2. For every clause C in F_{CNF} that contains both l and $\neg l$ do:
    1. $F_{CNF} \leftarrow remove\_from\_formula(C, F_{CNF})$
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    1. Return SAT
4. If F_{CNF} has empty clause then
    1. Return UNSAT
5. If $\nexists l$ that occurs with both polarities in different clauses in F_{CNF}
    1. Return SAT
6. Pick a literal l that occurs with both polarities in F_{CNF}.
    1. $F_{CNF} \leftarrow Resolution(C, l, F_{CNF})$
7. For every clause C that contains l or $\neg l$ do :
    1. $F_{CNF} \leftarrow remove\_from\_formula(C, F_{CNF})$

Can we do better?

Pure Literal Elimination

Pure literal: a literal l all of which occurrences in F have the same polarity.

Example: $(p \lor q \lor r) \land (\neg q \lor r) \land (p \lor \neg r) \land (p \lor \neg q)$

$$(\boxed{p} \lor q \lor r) \land (\neg q \lor r) \land (\boxed{p} \lor \neg r) \land (\boxed{p} \lor \neg q)$$

Literal p has positive polarity in all occurrence in F. P is pure literal.

$(p \lor \neg q \lor r) \land (\neg q \lor r) \land (\neg p \lor \neg r) \land (p \lor \neg q)$ — $\neg q$ is pure literal

Pure Literal Elimination

Pure literal: a literal l all of which occurrences in F have the same polarity.

For every clause that contains a pure literal:

$$F_{CNF} \leftarrow remove\_from\_formula(C, F_{CNF})$$

DP algorithm for SAT Solving (Martin Davis - Hilary Putnam 1960)

1. Start with F_{CNF}
2. For every clause C in F_{CNF} that either contains both l and $\neg l$ or has pure literal do:
    1. $F_{CNF} \leftarrow remove\_from\_formula(C, F_{CNF})$
3. If F_{CNF} is empty
    1. Return SAT
4. If F_{CNF} has empty clause then
    1. Return UNSAT
5. If $\nexists l$ that occurs with both polarities in different clauses in F_{CNF}
    1. Return SAT
6. Pick a literal l that occurs with both polarities in F_{CNF}.
    1. $F_{CNF} \leftarrow Resolution(C, l, F_{CNF})$
7. For every clause C that contains l or $\neg l$ do :
    1. $F_{CNF} \leftarrow remove\_from\_formula(C, F_{CNF})$

DP algorithm for SAT Solving (Martin Davis - Hilary Putnam 1960)

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3. If F_{CNF} is empty
    1. Return SAT
4. If F_{CNF} has empty clause then
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5. Pick a literal l that occurs with both polarities in F_{CNF}.
    1. $F_{CNF} \leftarrow Resolution(C, l, F_{CNF})$
6. For every clause C that contains l or $\neg l$ do :
    1. $F_{CNF} \leftarrow remove\_from\_formula(C, F_{CNF})$

DP algorithm

$$F = (p \vee q) \wedge (p \vee \neg q) \wedge (\neg p \vee r) \wedge (\neg p \vee \neg r)$$

* No pure literal, no clause with $l \vee \neg l$

Pick literal p

$$(q \vee r) \;\wedge\; (q \vee \neg r) \;\wedge\; (\neg q \vee r) \;\wedge\; (\neg q \vee \neg r)$$

* No pure literal, no clause with $l \vee \neg l$

Pick literal q

$$(r) \;\wedge\; (r \vee \neg r) \;\wedge\; (\neg r \vee r) \;\wedge\; (\neg r)$$

*remove clauses with $l \vee \neg l$

$$(r) \wedge (\neg r)$$

Pick literal r

$$\square$$

F has empty clause — UNSAT

DP algorithm

$$F = (p \vee q \vee r) \wedge (q \vee \neg r \vee \neg s) \wedge (\neg q \vee s) \wedge (\neg p \vee \neg s)$$

Course Webpage

Thanks!