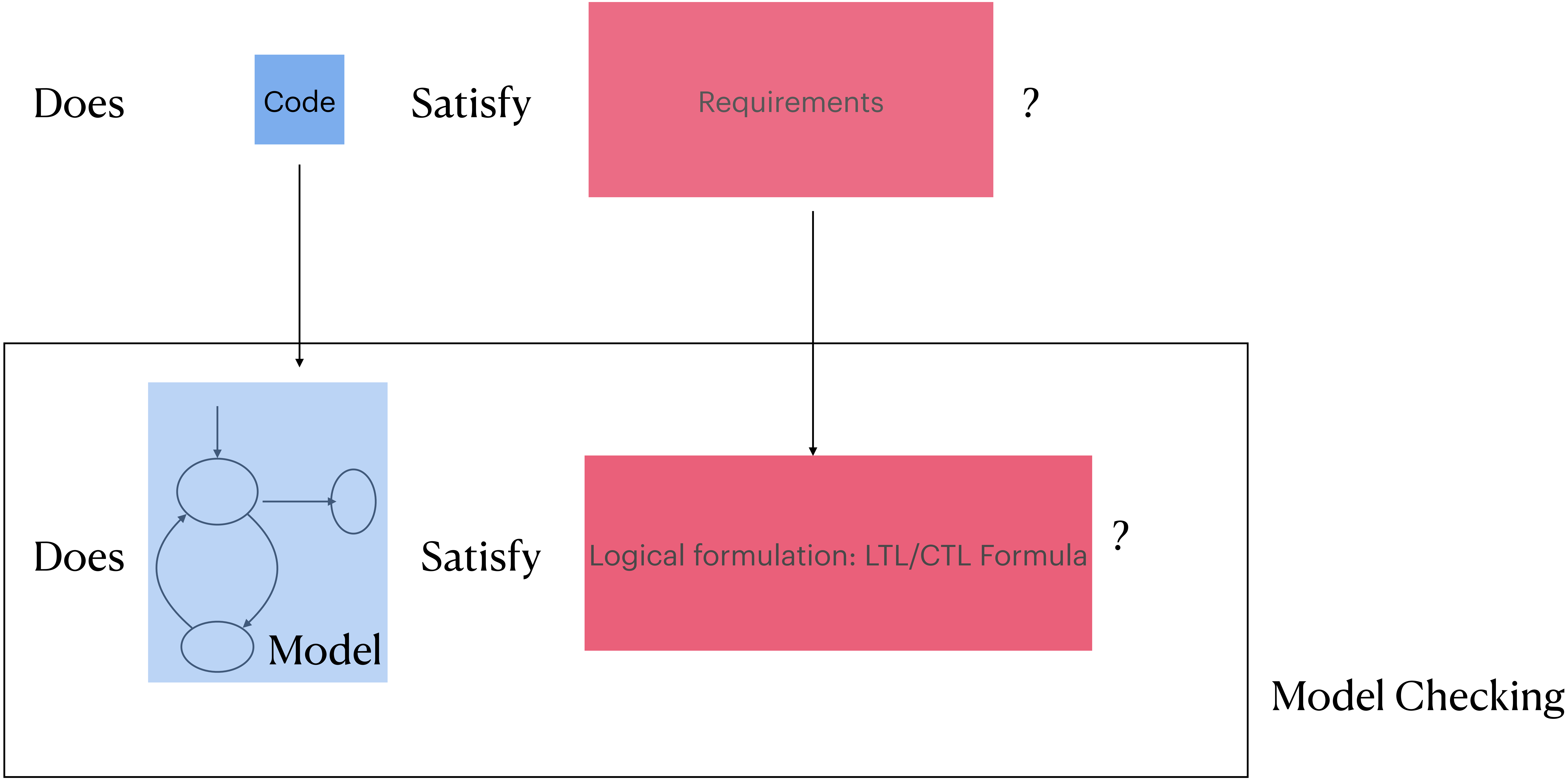
COL:750

Foundations of Automatic Verification

Instructor: Priyanka Golia

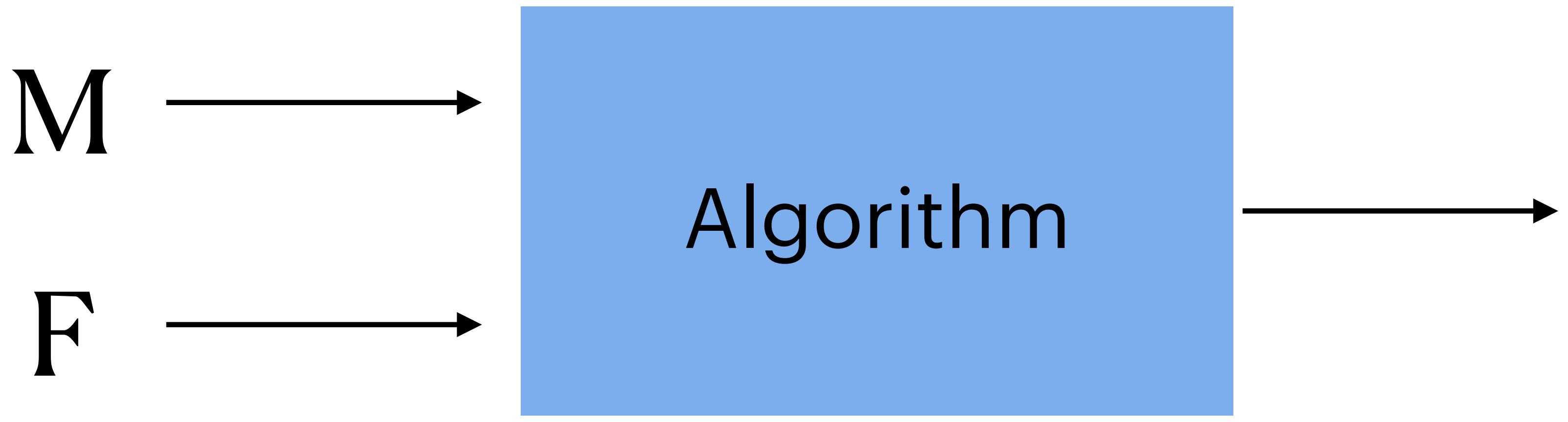
Course Webpage

https://priyanka-golia.github.io/teaching/COL-750/index.html


Does
Code
Satisfy
Requirements
?
Does
Model
Satisfy
Logical formulation: LTL/CTL Formula
?
Model Checking

Model Checking Algorithm

$$M, s \vDash F?$$



$S' \quad s.t.\, S' \subseteq S$, and $M, s \vDash F \ \forall s \in S'$

All states s of the model M that satisfy F

Note that not necessarily $I \subseteq S'$

Labelling Algorithm —

1. Does not scale well to large systems due to state explosion.
2. Memory-intensive as it maintains explicit labels for each state.

We need better data structure.

CTL Model Checking Algorithm — BDD basedAlgorithm

1. Input — a Model M, and a CTL formula F.
2. Output — S' (the set of states of M that satisfy the formula F.)

BDD — Binary Decision Diagrams.

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A binary decision diagram (BDD) is a data structure that is used to represent a Boolean function. $\{x, y, z, \ldots, \} \rightarrow \{0,1\}$

BDDs can be considered as a compressed representation of sets or relations.

$$F = (x \land y) \lor (\neg y \land z)$$

X	Y	Z	F
0	0	0	0
0	0	1	1
0	1	0	0
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1	0	0	0
1	0	1	1
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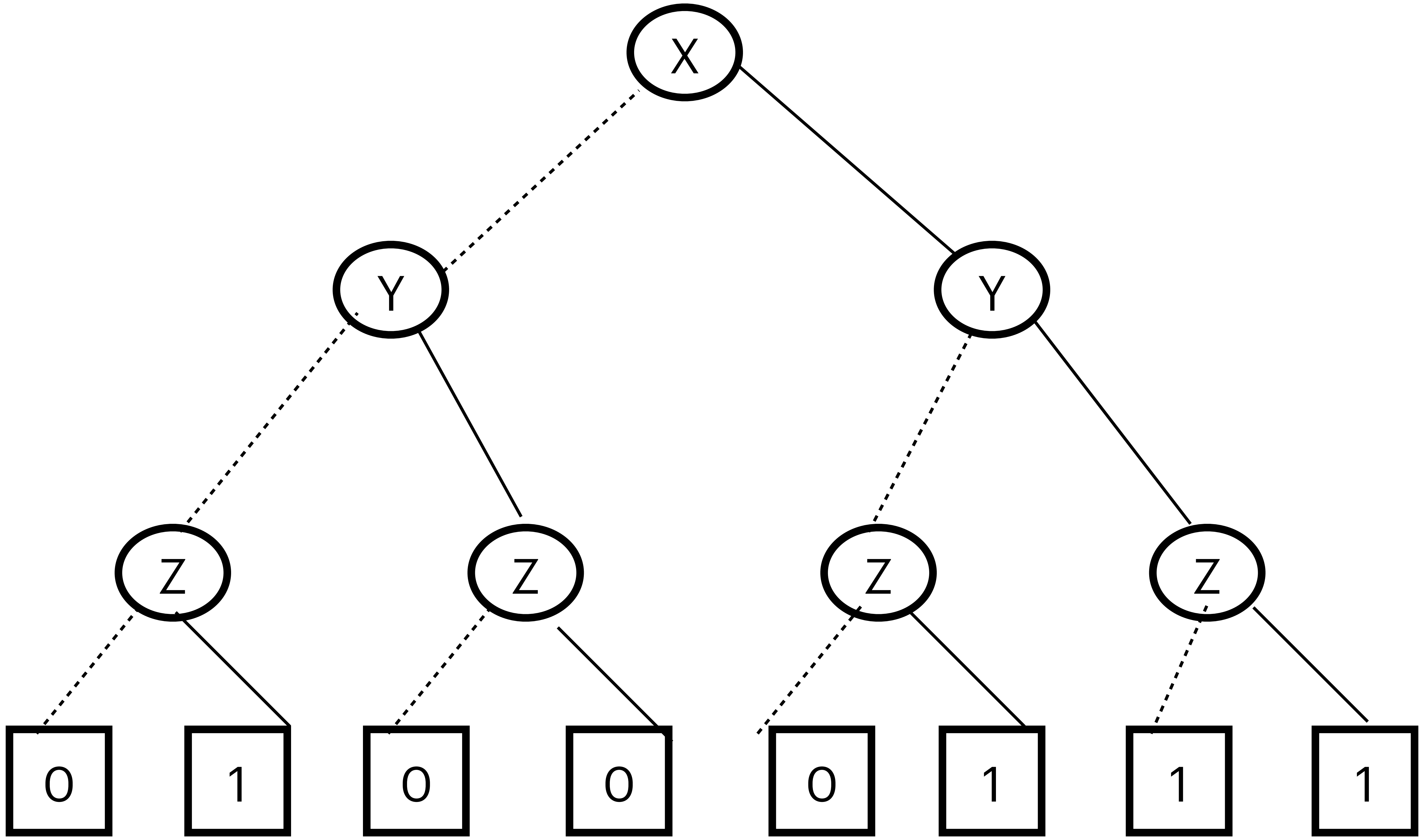

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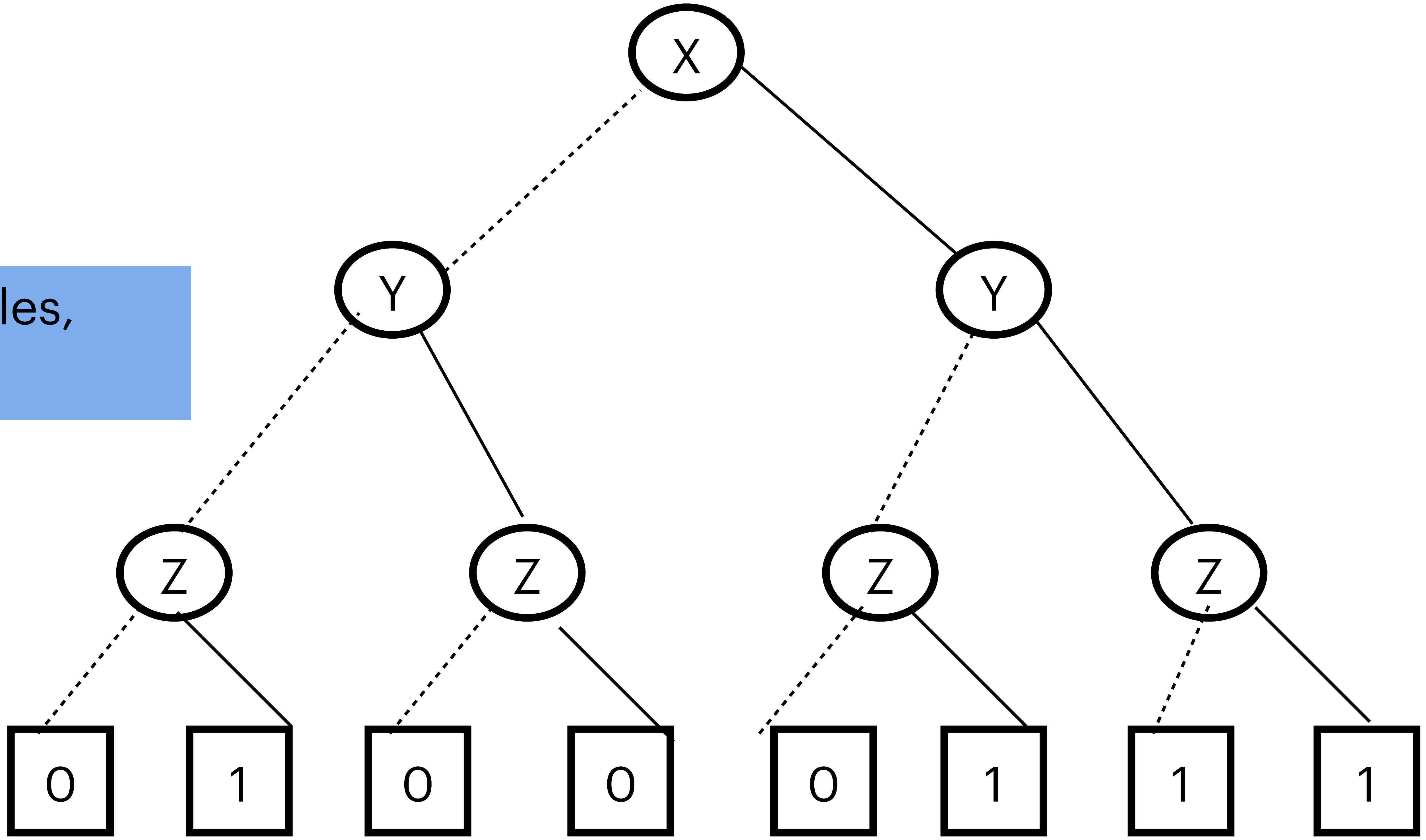
Space: for n variables, $O(2^{n+1} - 1)$



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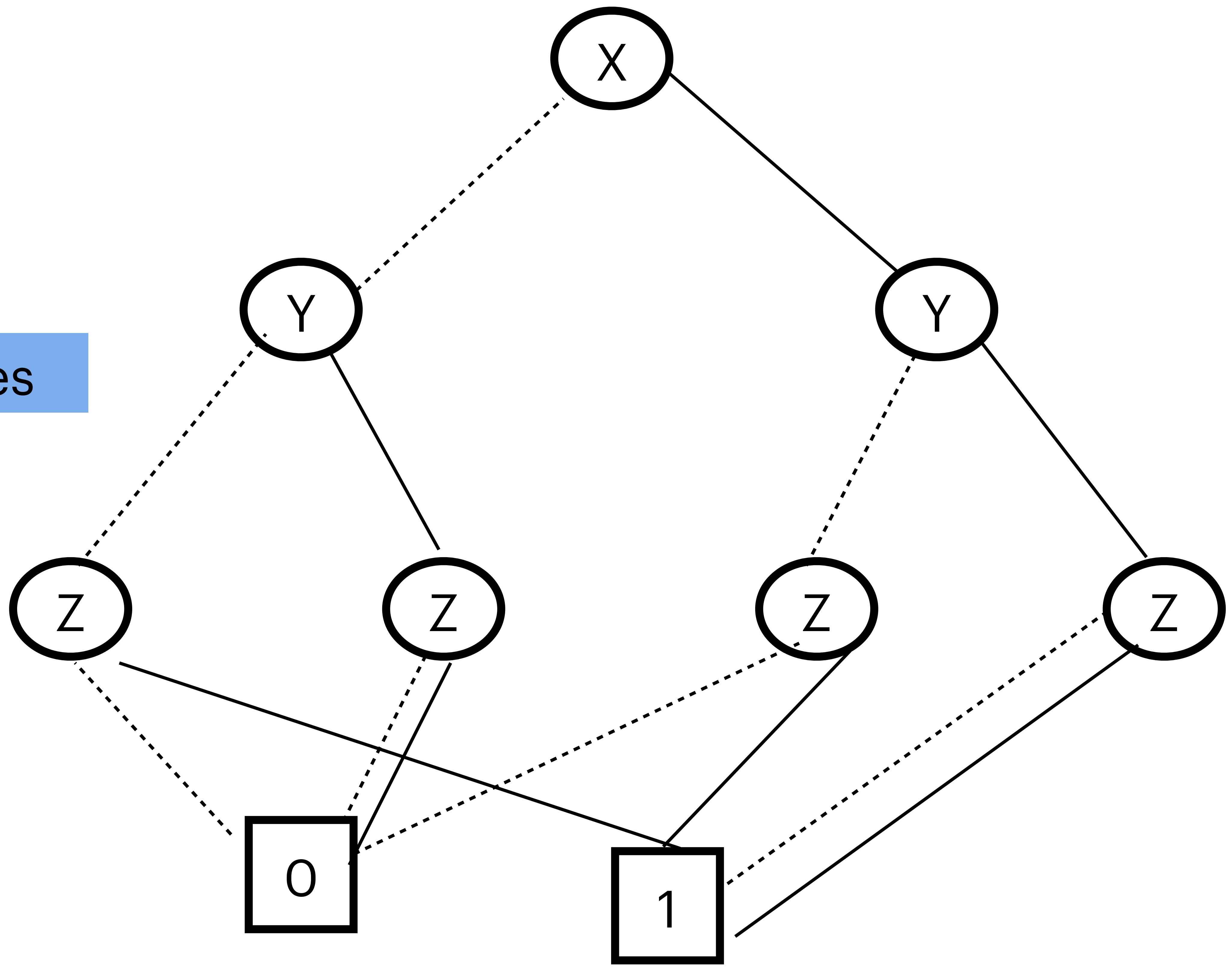
Removal of duplicate leaves



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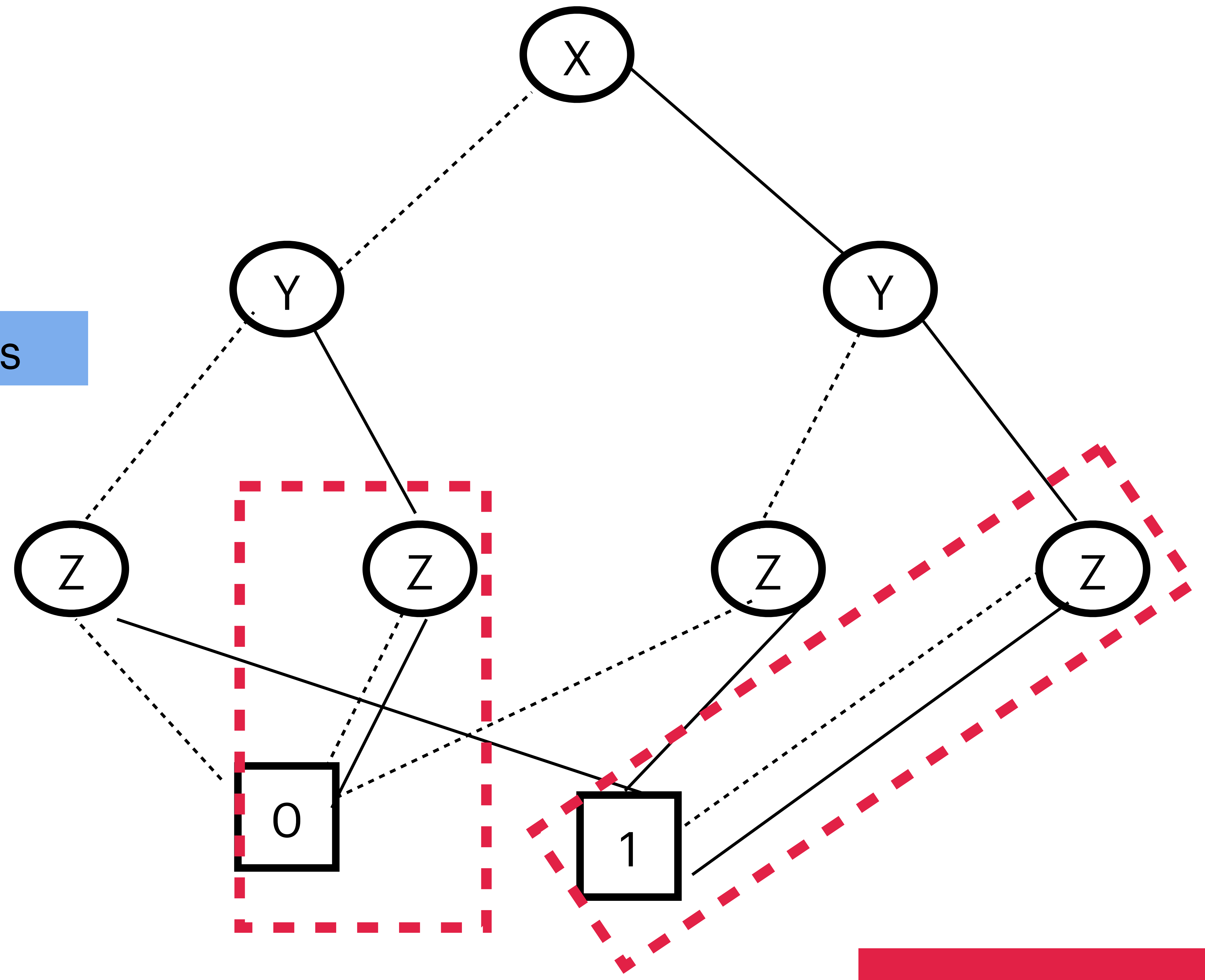
Removal of duplicate tests



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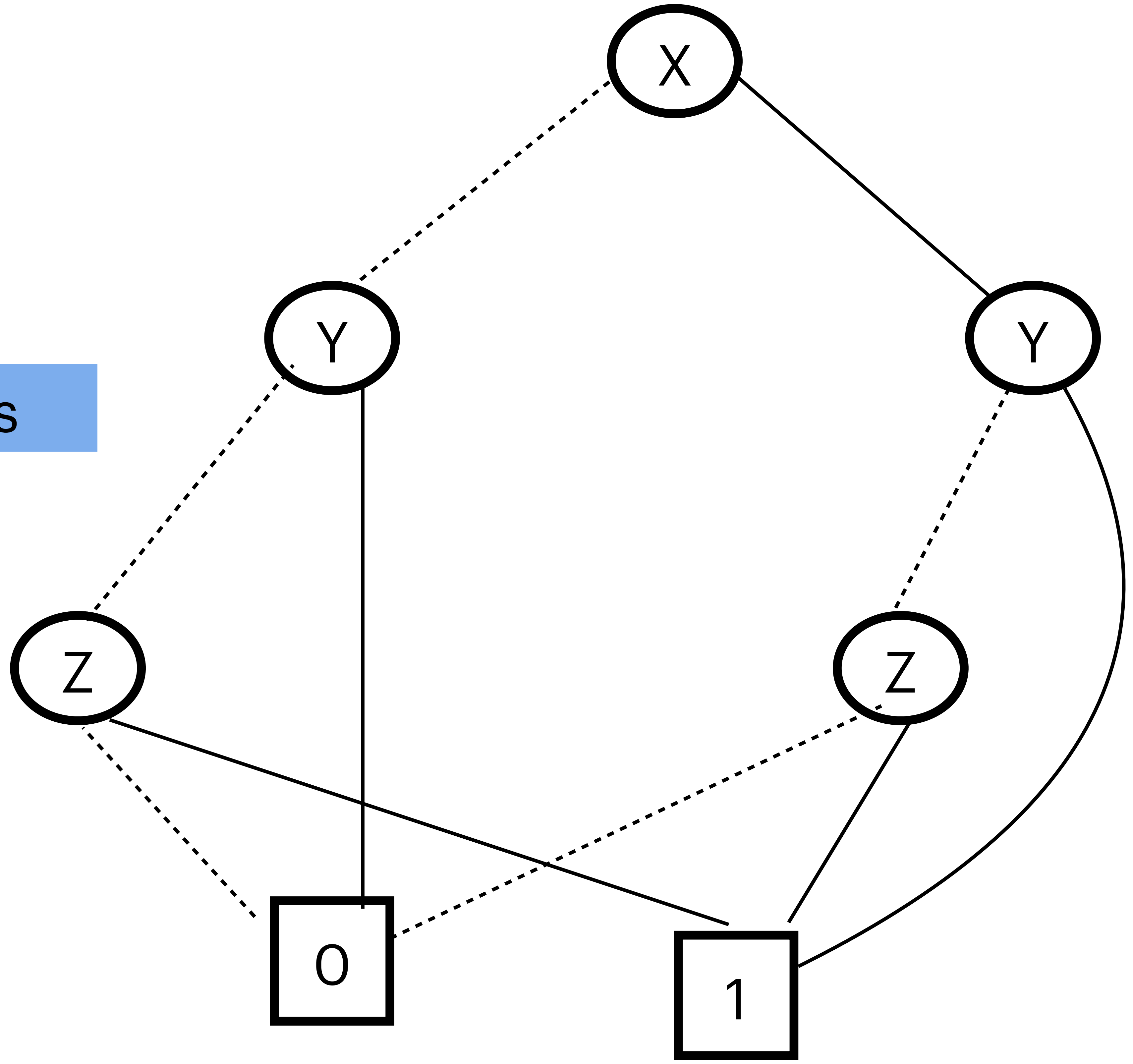

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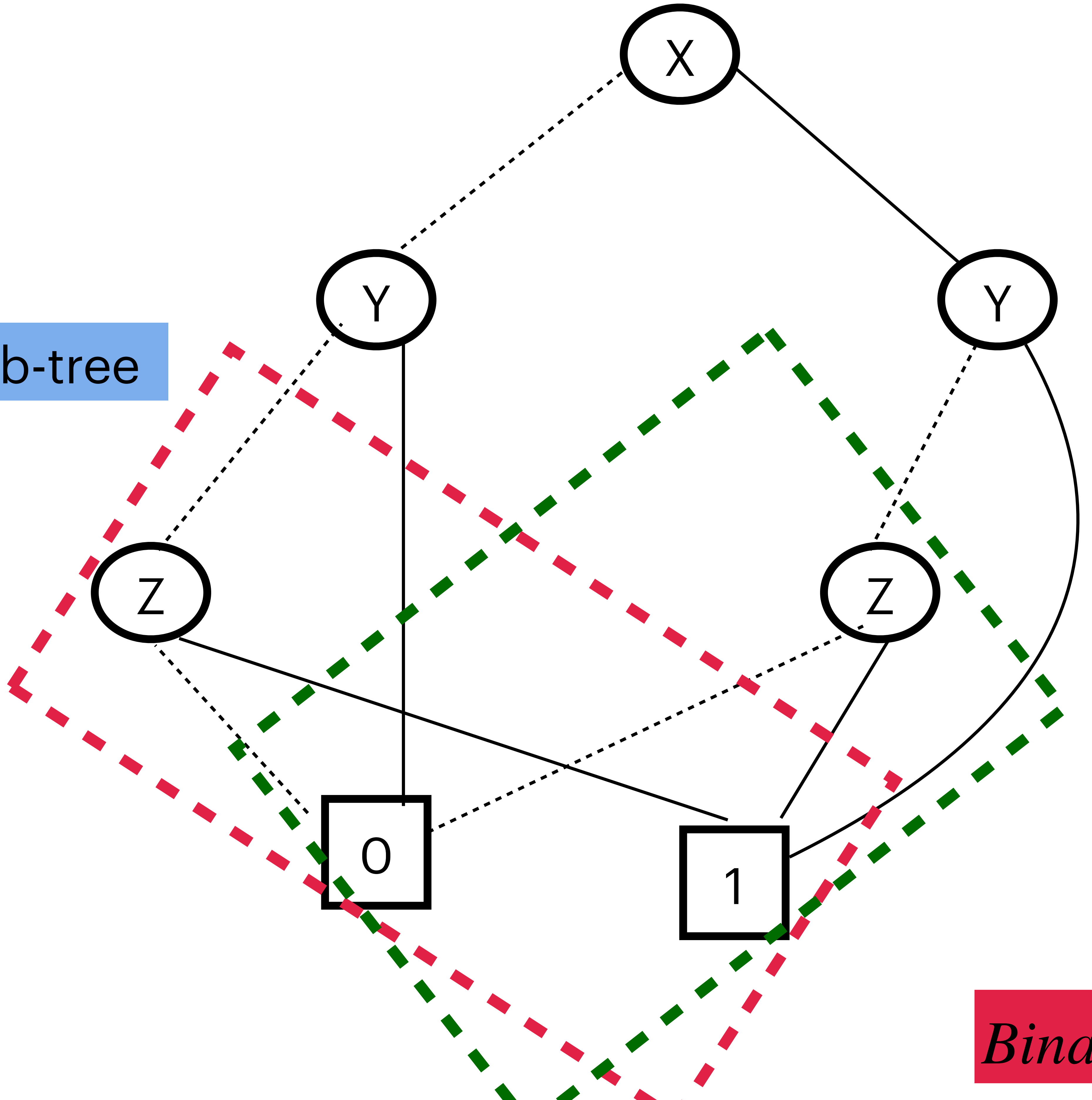
Removal of duplicate sub-tree



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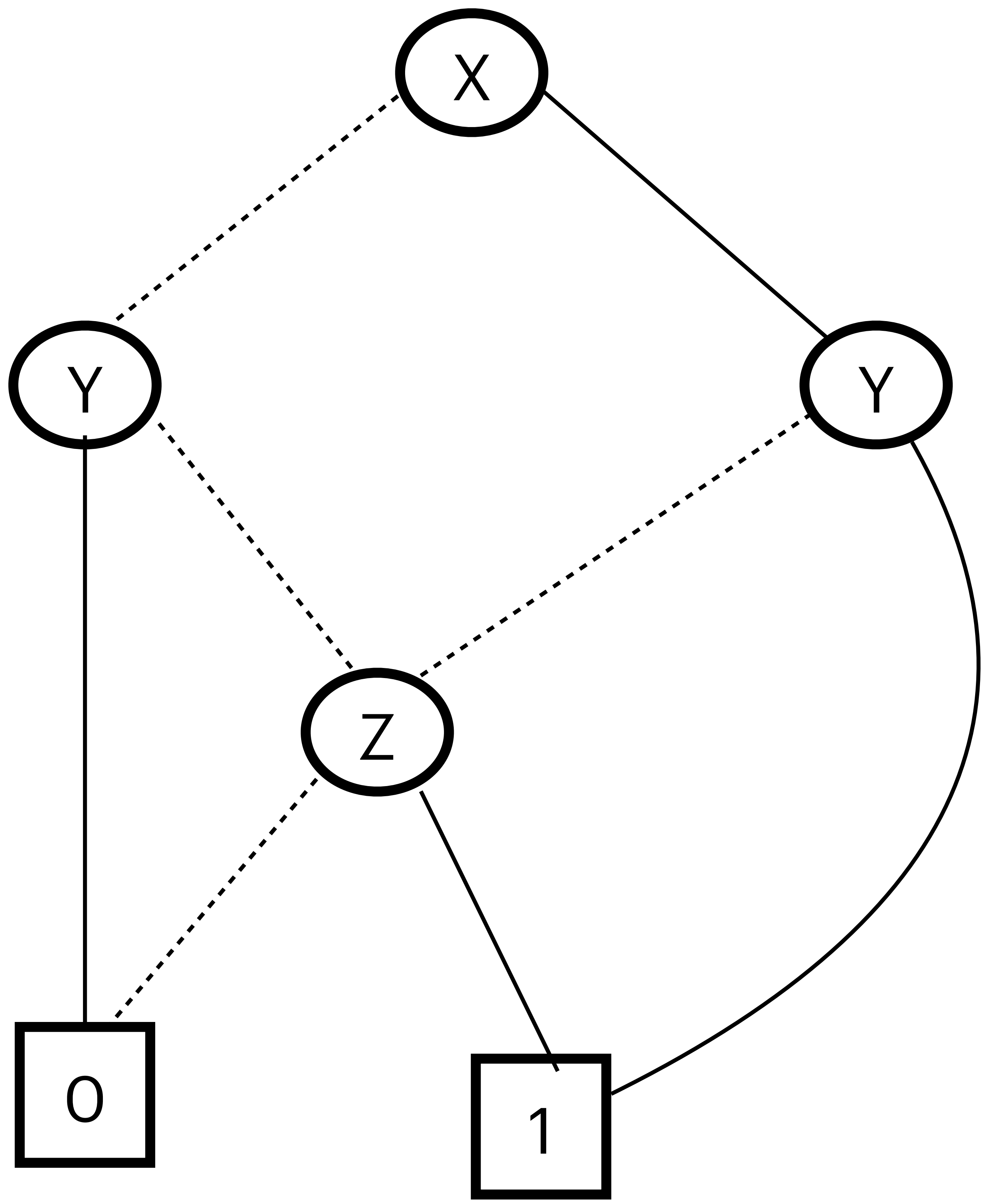
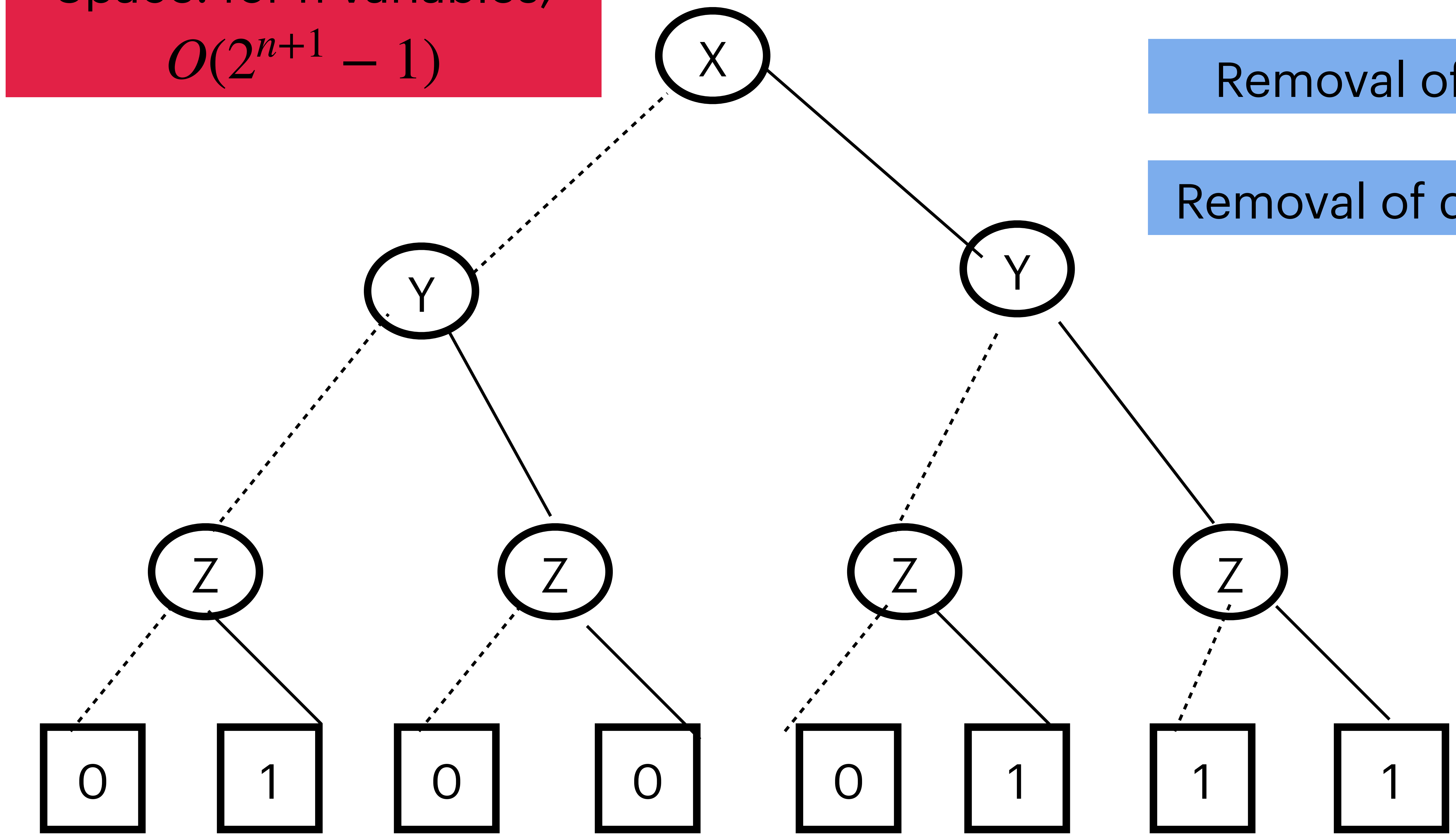


Binary Decision Diagram

RBDD — Reduced Binary Decision Diagrams

$$F = (x \wedge y) \vee (\neg y \wedge z)$$

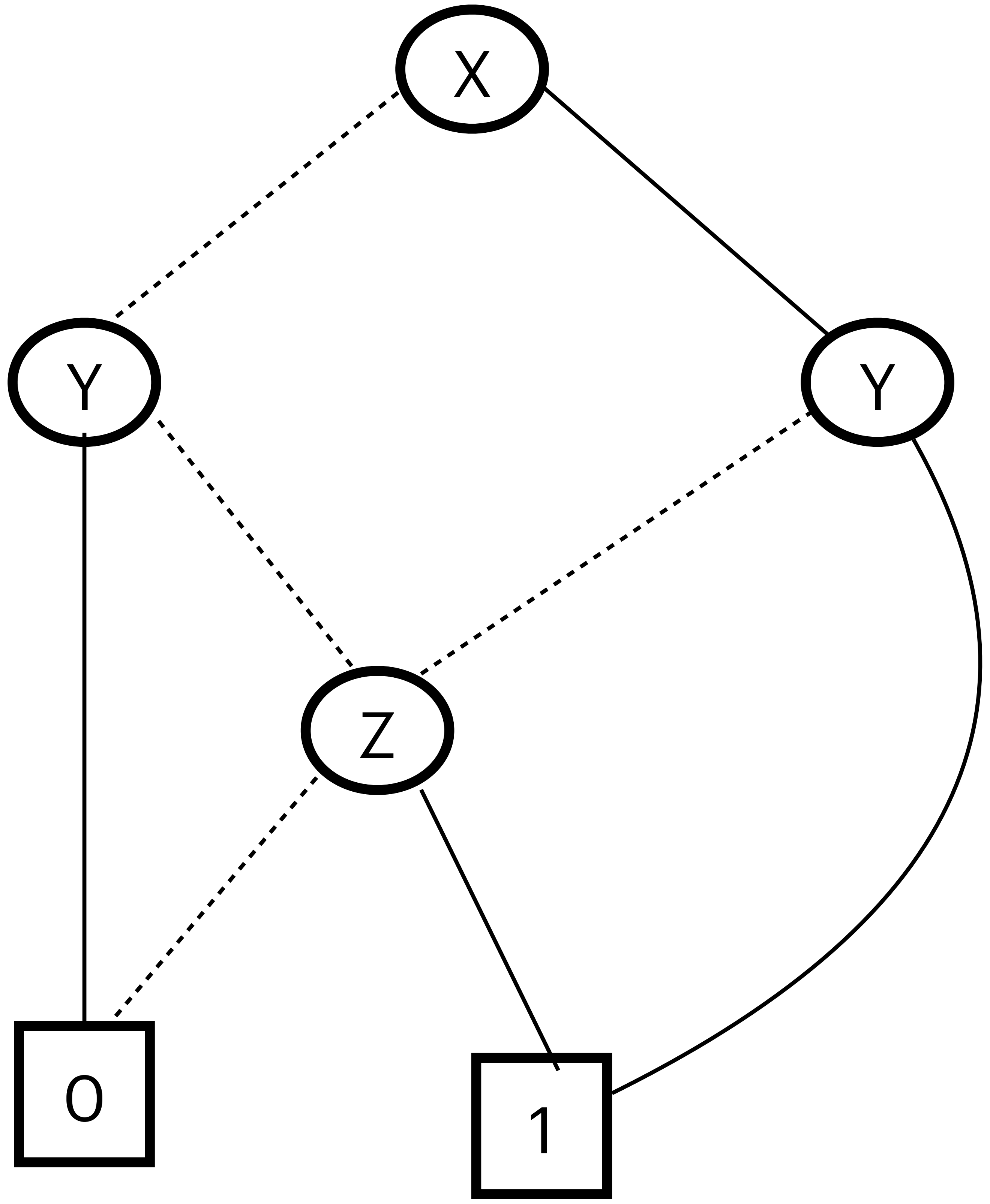
Space: for n variables, $O(2^{n+1} - 1)$



Removal of duplicate leaves

Removal of duplicate tests

Removal of duplicate sub-tree



BDD — Binary Decision Diagrams

$$F(x_1, x_2, x_3, x_4) = \begin{cases} 1 \text{ — if even number of variables are 1} \\ 0 \text{ — otherwise} \end{cases}$$

Create a ROBDD.

Assuming order to be x_1, x_2, x_3, x_4

ROBDD — Reduced Ordered Binary Decision Diagrams

$$F(x_1, x_2, x_3, x_4) = \begin{cases} 1 \text{ — if even number of variables are 1} \\ 0 \text{ — otherwise} \end{cases}$$

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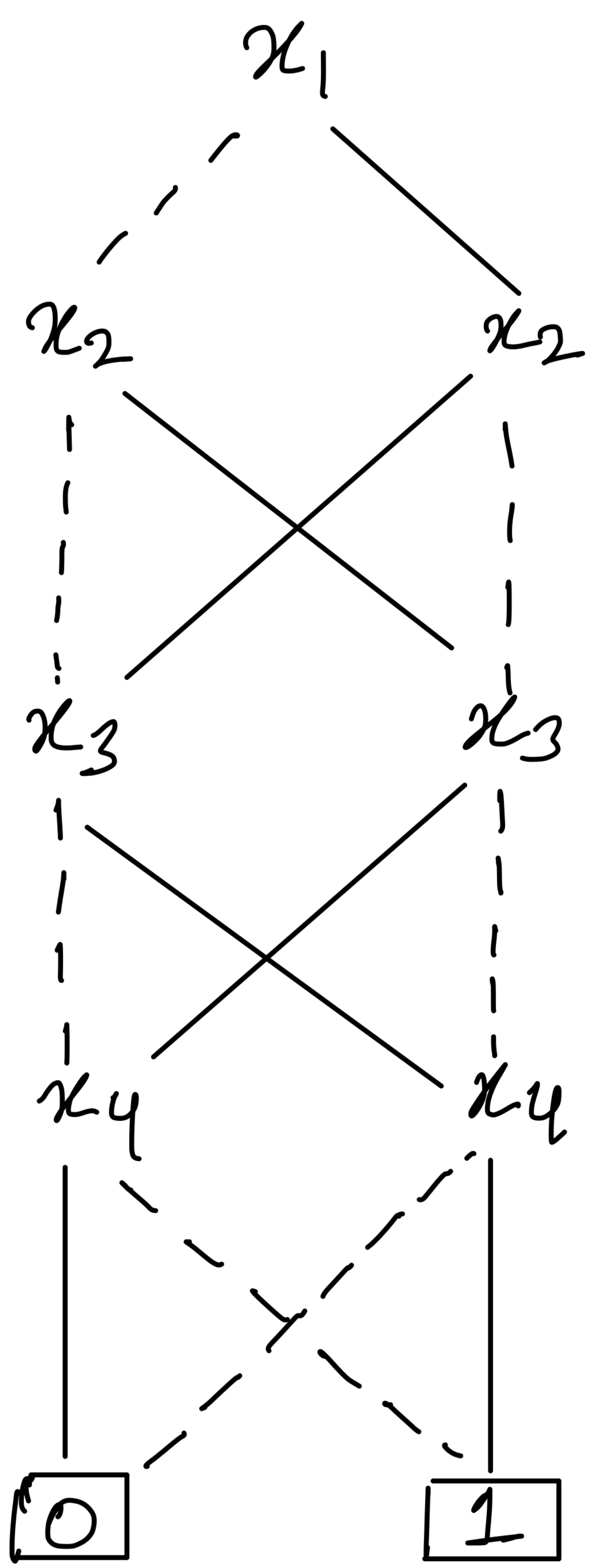

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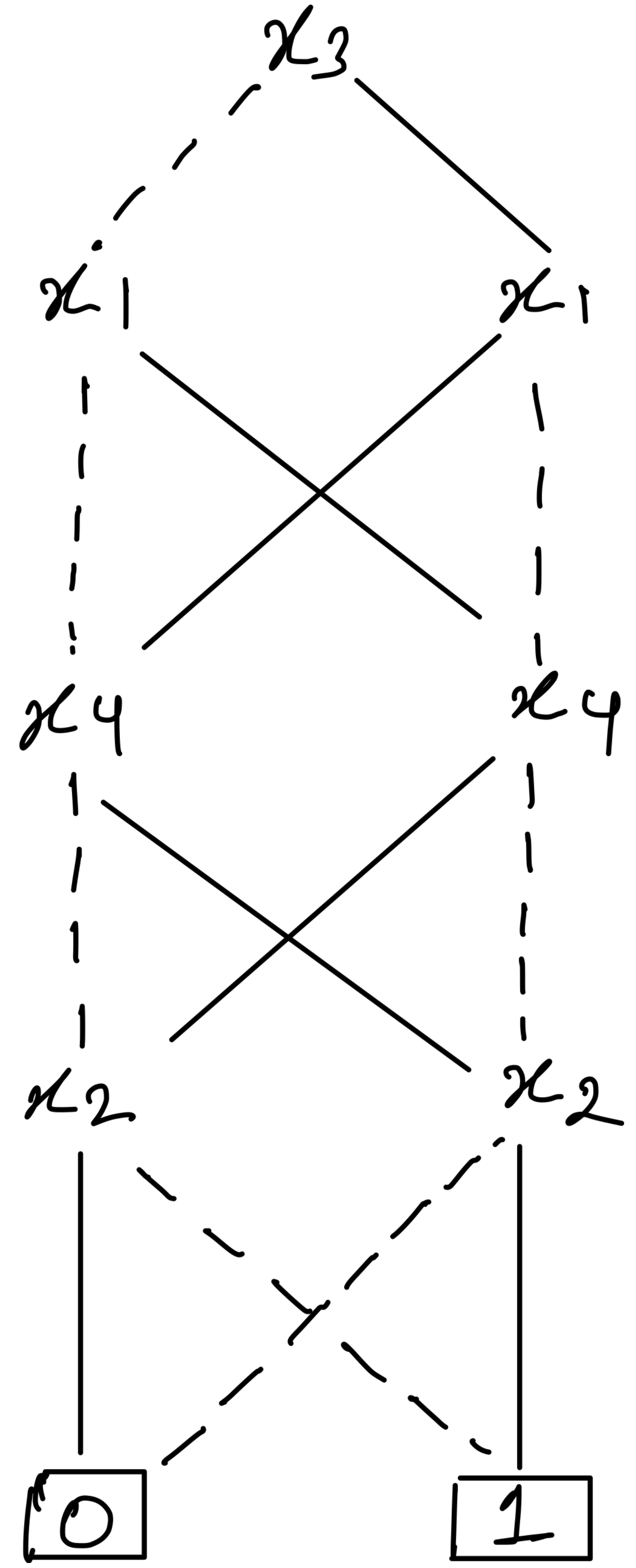
$$F(x_1, x_2, x_3, x_4) = \begin{cases} 1 \text{ — if even number of variables are 1} \\ 0 \text{ — otherwise} \end{cases}$$

Create a ROBDD.

Assuming order to be x_3, x_1, x_4, x_2

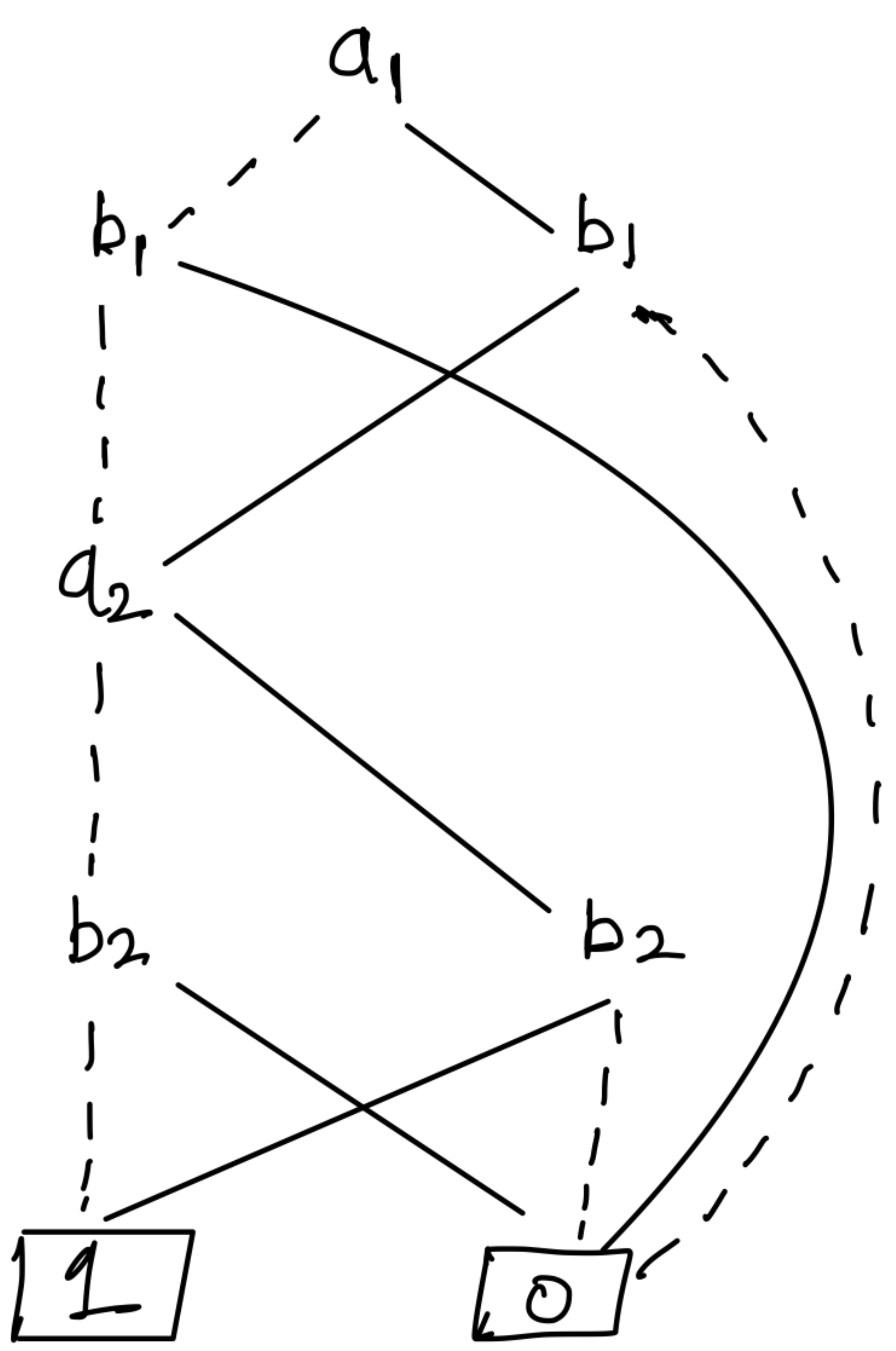


ROBDD — Reduced Ordered Binary Decision Diagrams

$$F = (a_1 \leftrightarrow b_1) \land (a_2 \leftrightarrow b_2)$$

Create a ROBDD.

Assuming order to be a_1, b_1, a_2, b_2



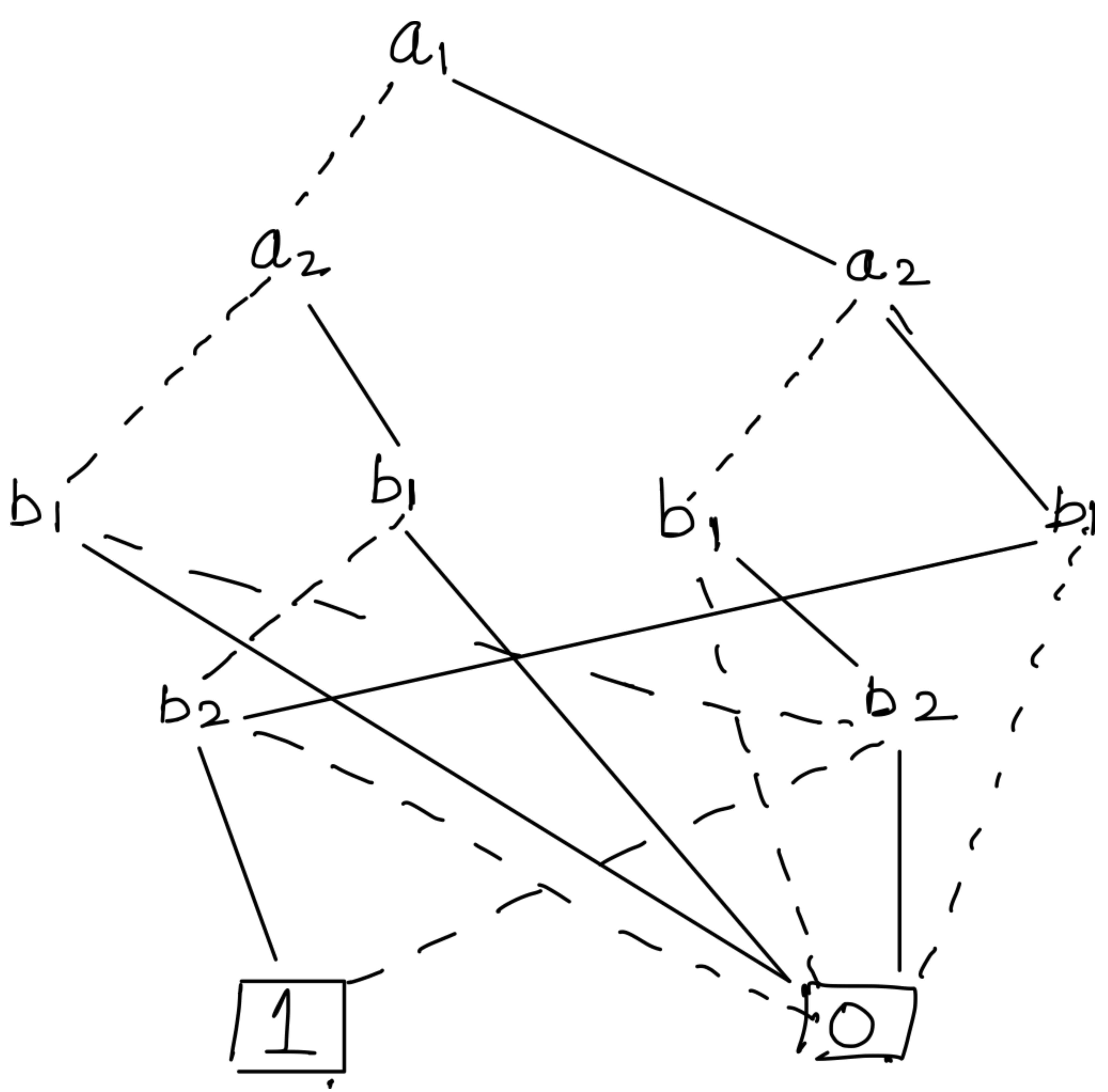
Number of nodes $2n + n + 2$

ROBDD — Reduced Ordered Binary Decision Diagrams

$F = (a_1 \leftrightarrow b_1) \wedge (a_2 \leftrightarrow b_2)$

Create a ROBDD.

Assuming order to be a_1, a_2, b_1, b_2



Number of nodes $3 \times 2^n - 1$

ROBDD — Reduced Ordered Binary Decision Diagrams

For an n-bit comparator:

if we use the ordering $< a_1, b_1, a_2, b_2, \ldots, a_n, b_n >$, the number of vertices will be $3n + 2$.

if we use the ordering $< a_1, a_2, \ldots, a_n, b_1, b_2 \ldots, b_n >$, the number of vertices is $3 \times 2^n - 1$.

Moreover, there are boolean functions that have exponential size OBDDs for any variable ordering.

An example is the middle output (nth output) of a combinational circuit to multiply two n bit integers

Given an order, ROBDD is always unique

ROBDD Operations

Assuming two ROBDDs over same variable ordering.

Given argument functions f and g, and a binary operator ,

- **APPLY** returns the function F <op> G.
- Works by traversing the argument graphs depth first.

Expanding for any variable x

$$F < op > G = \neg x(F|_{x=0} < op > G|_{x=0}) + x(F|_{x=1} < op > G|_{x=1})$$

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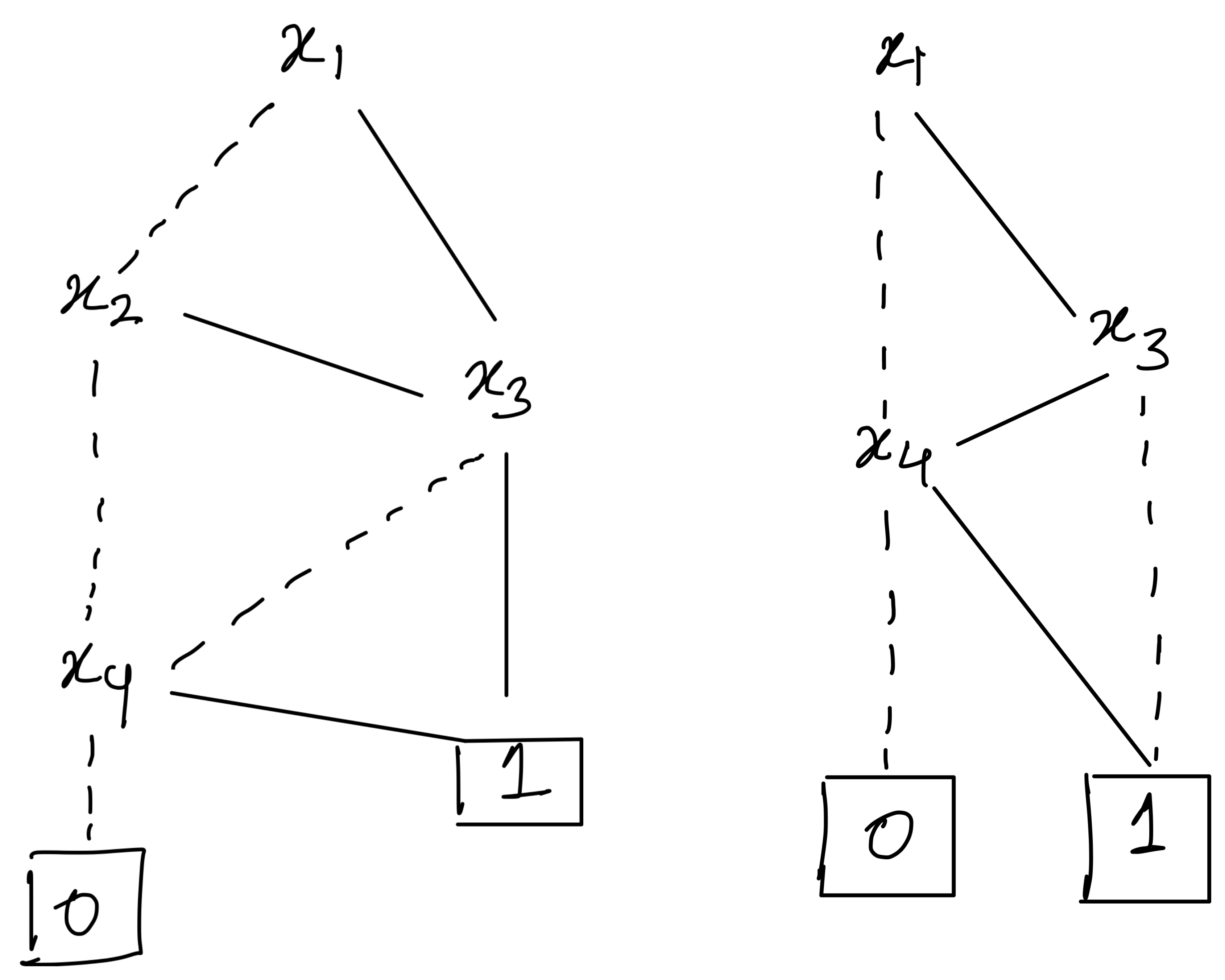
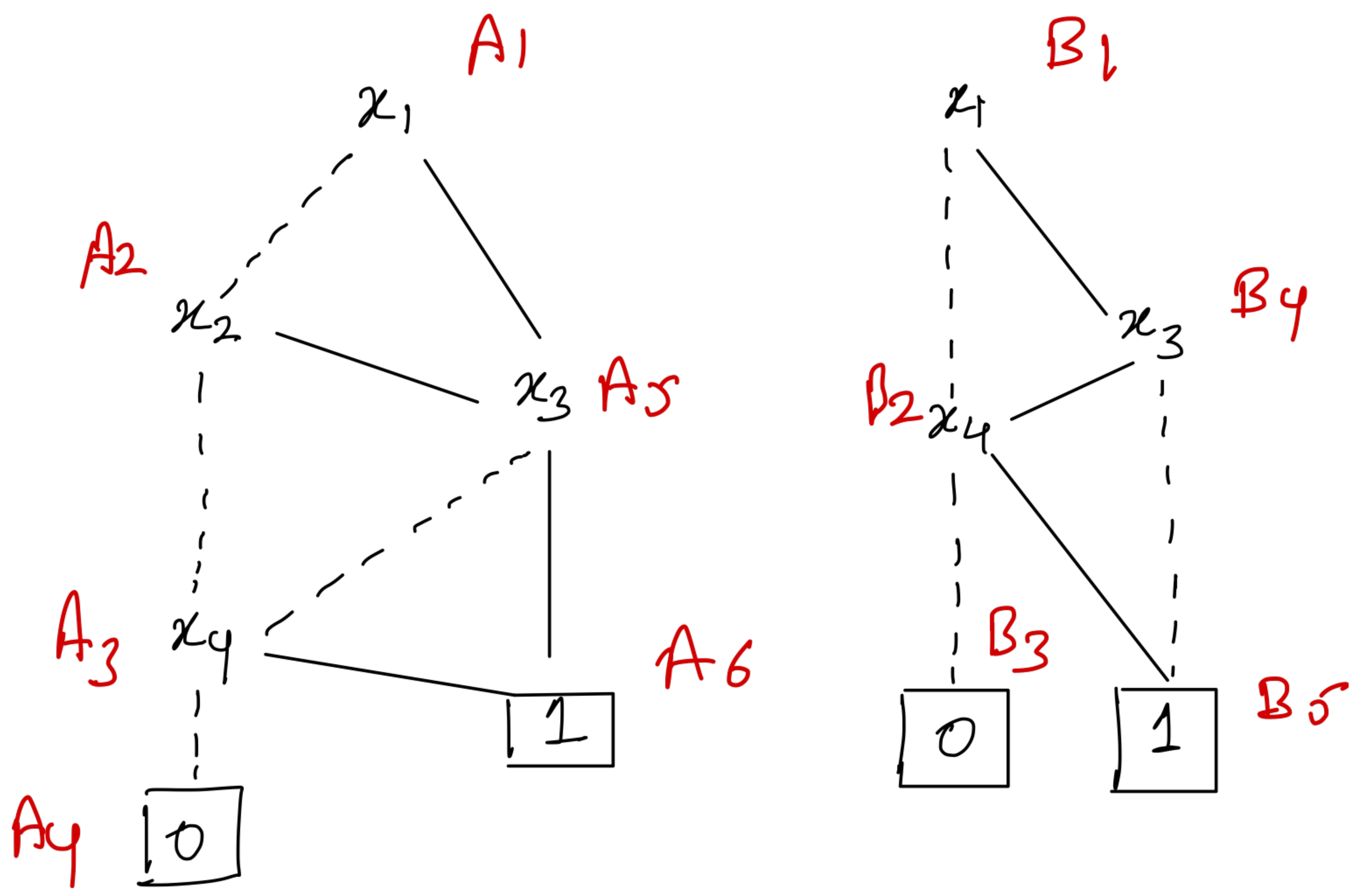
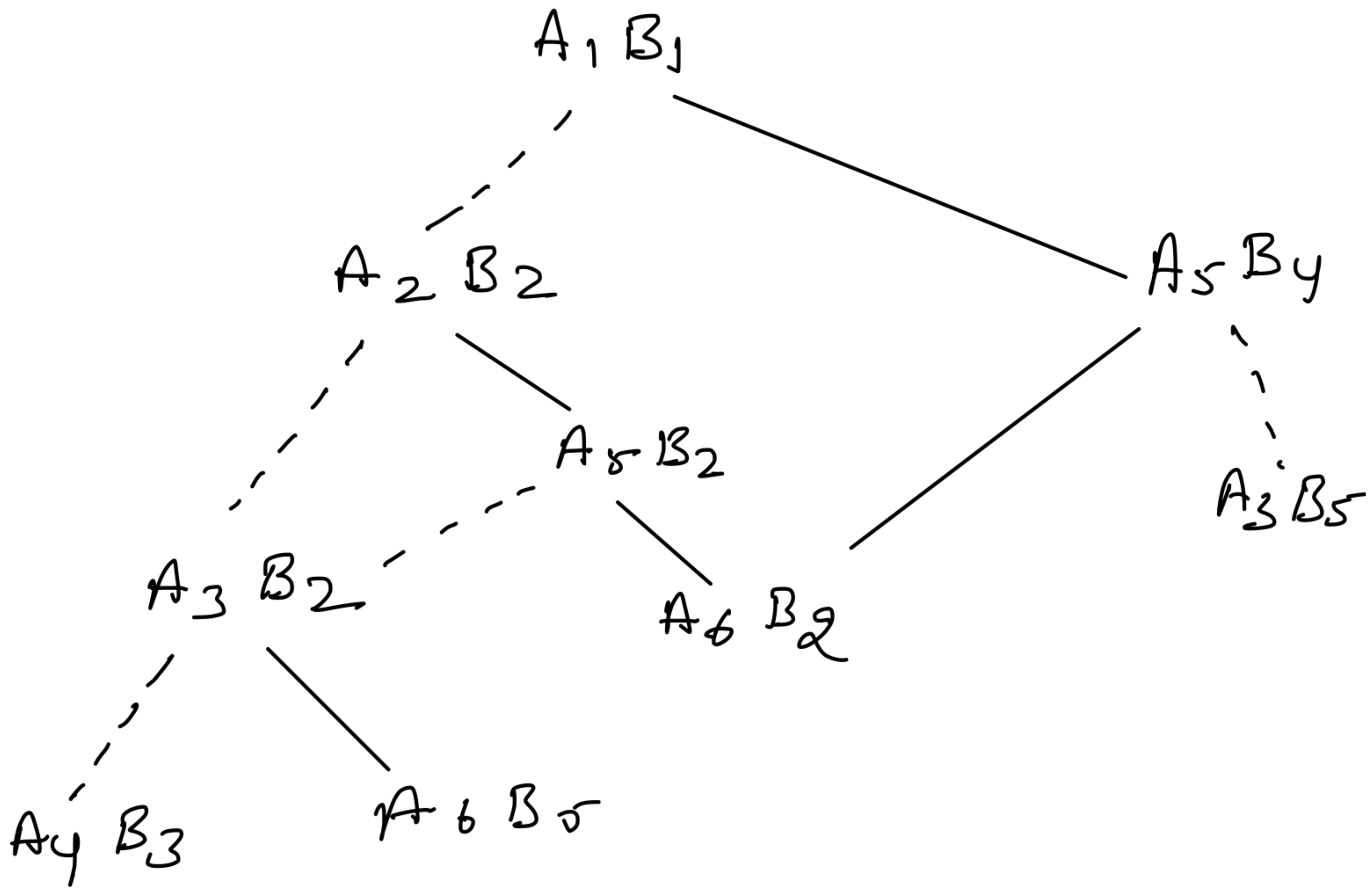


$F(x_1, x_2, x_3, x_4)$ $G(x_1, x_2, x_3, x_4)$

How about $F \vee G$?

ROBDD Operations (Apply)



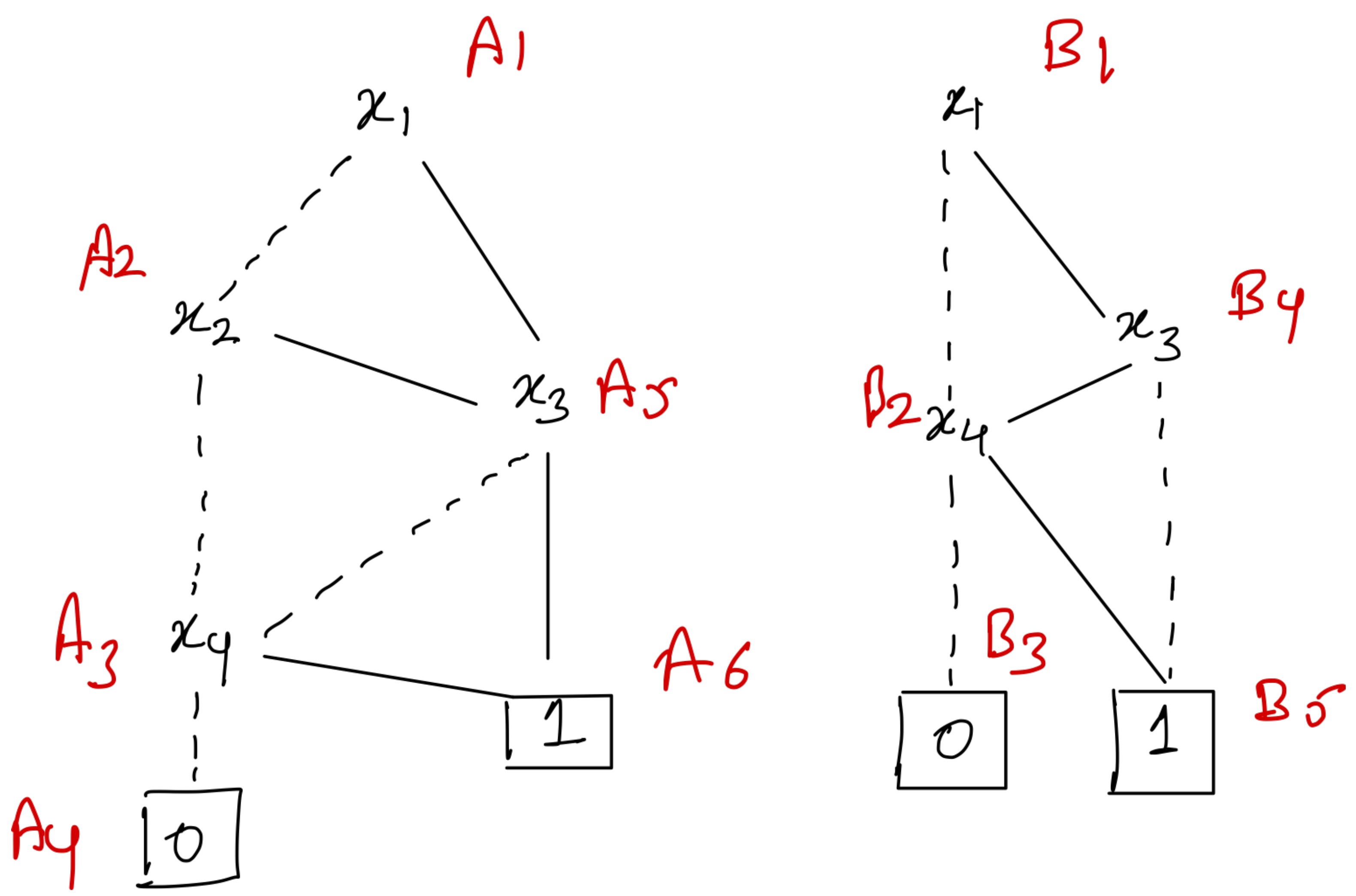


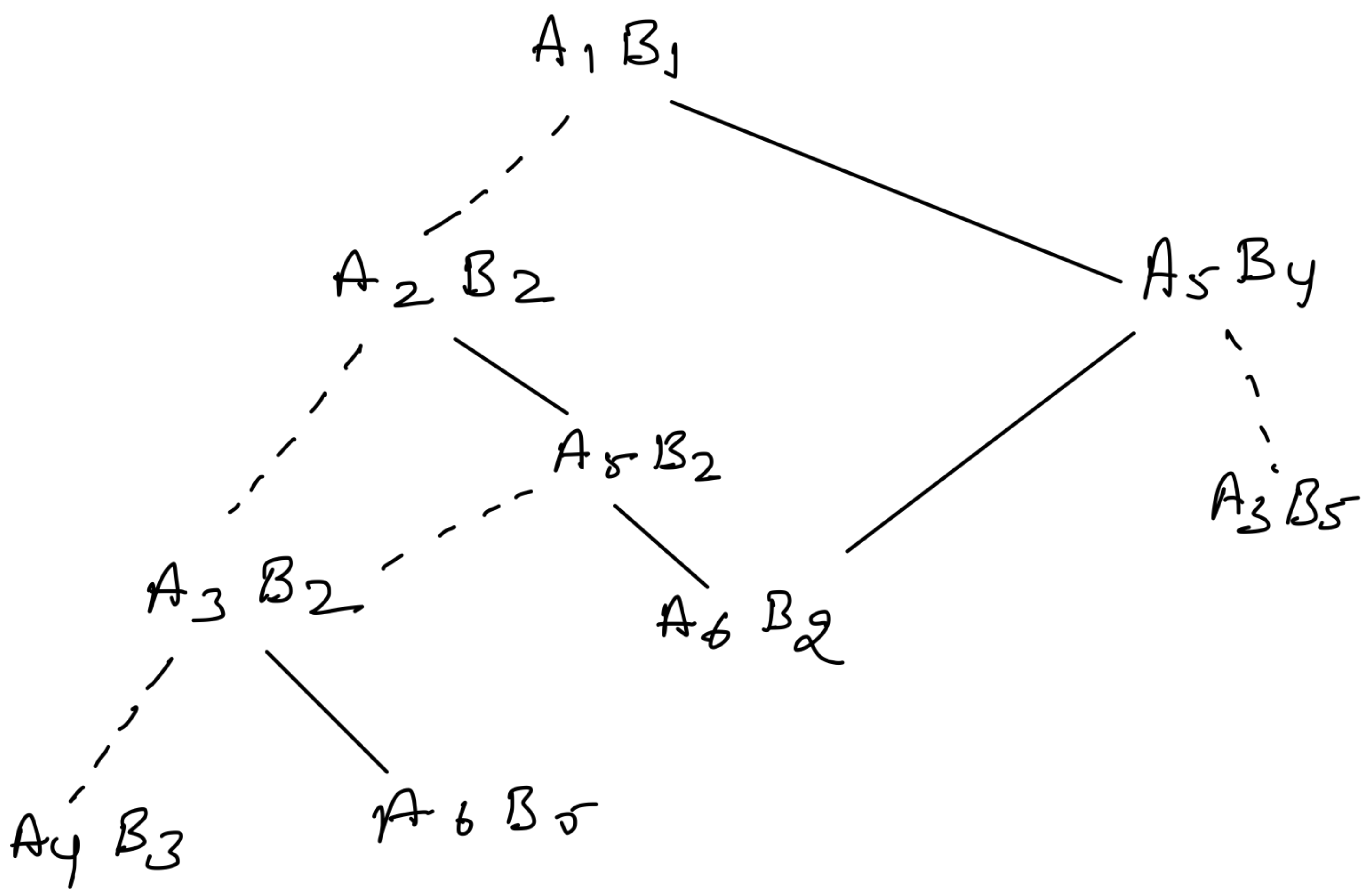
1. Depth first search — respect the ordering.

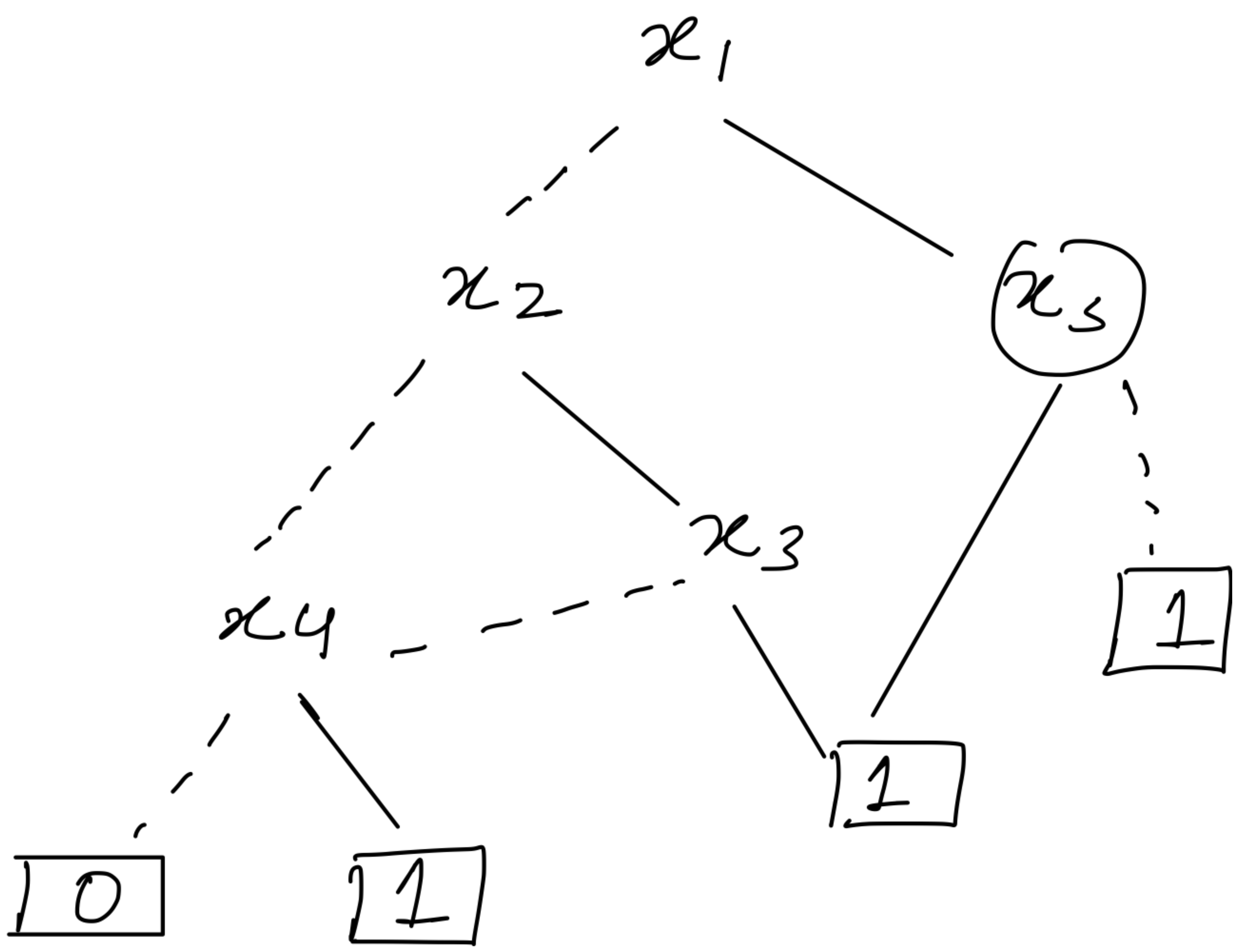
2. Reaching a terminal with a dominant value (e.g 1 for OR, 0 for AND) terminates recursion and returns an appropriately labeled terminal

3. Avoid multiple recursive calls on the same pair of arguments by a hash table

ROBDD Operations (Apply)







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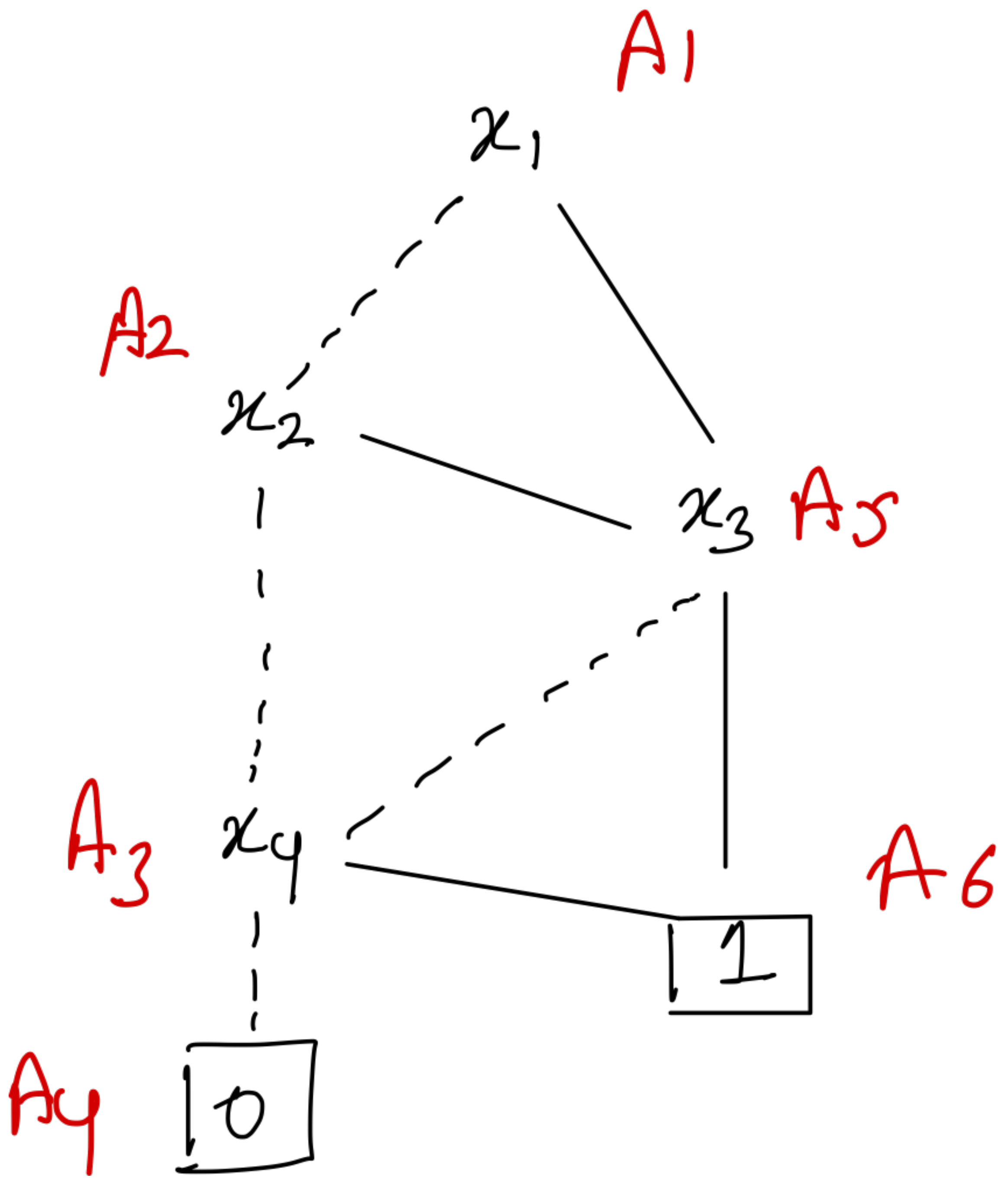
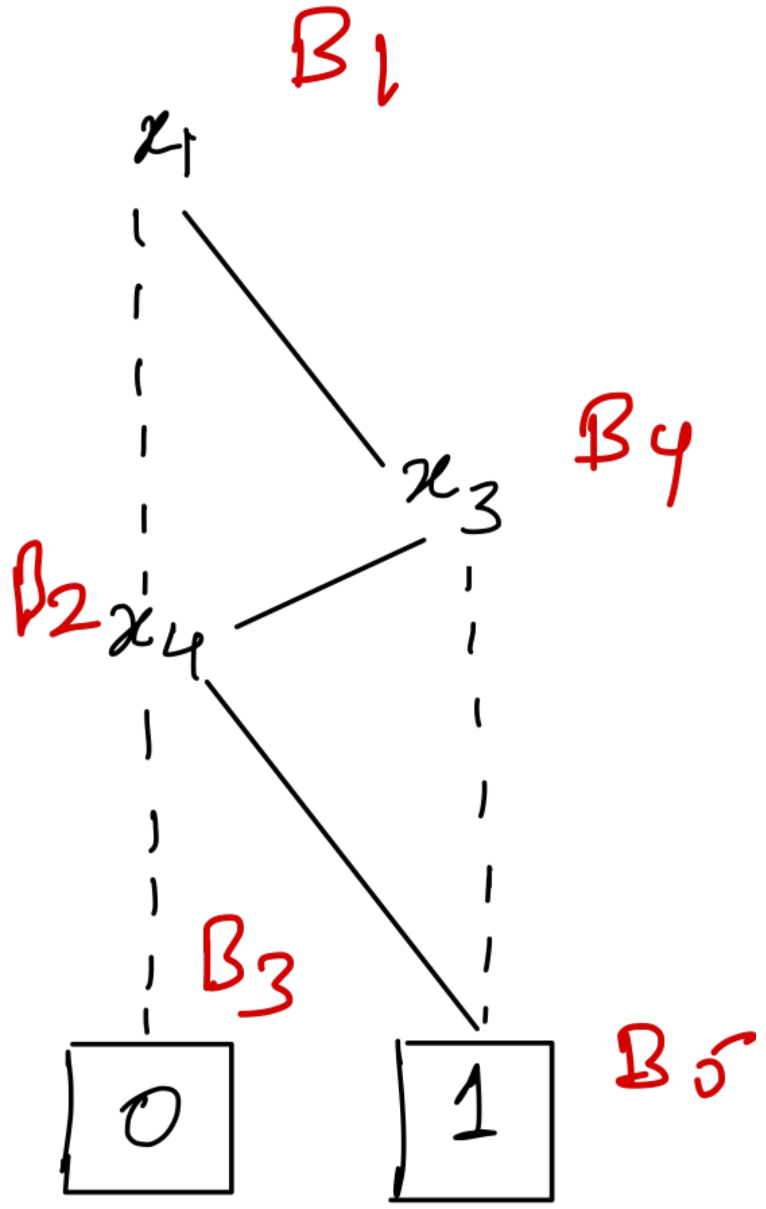
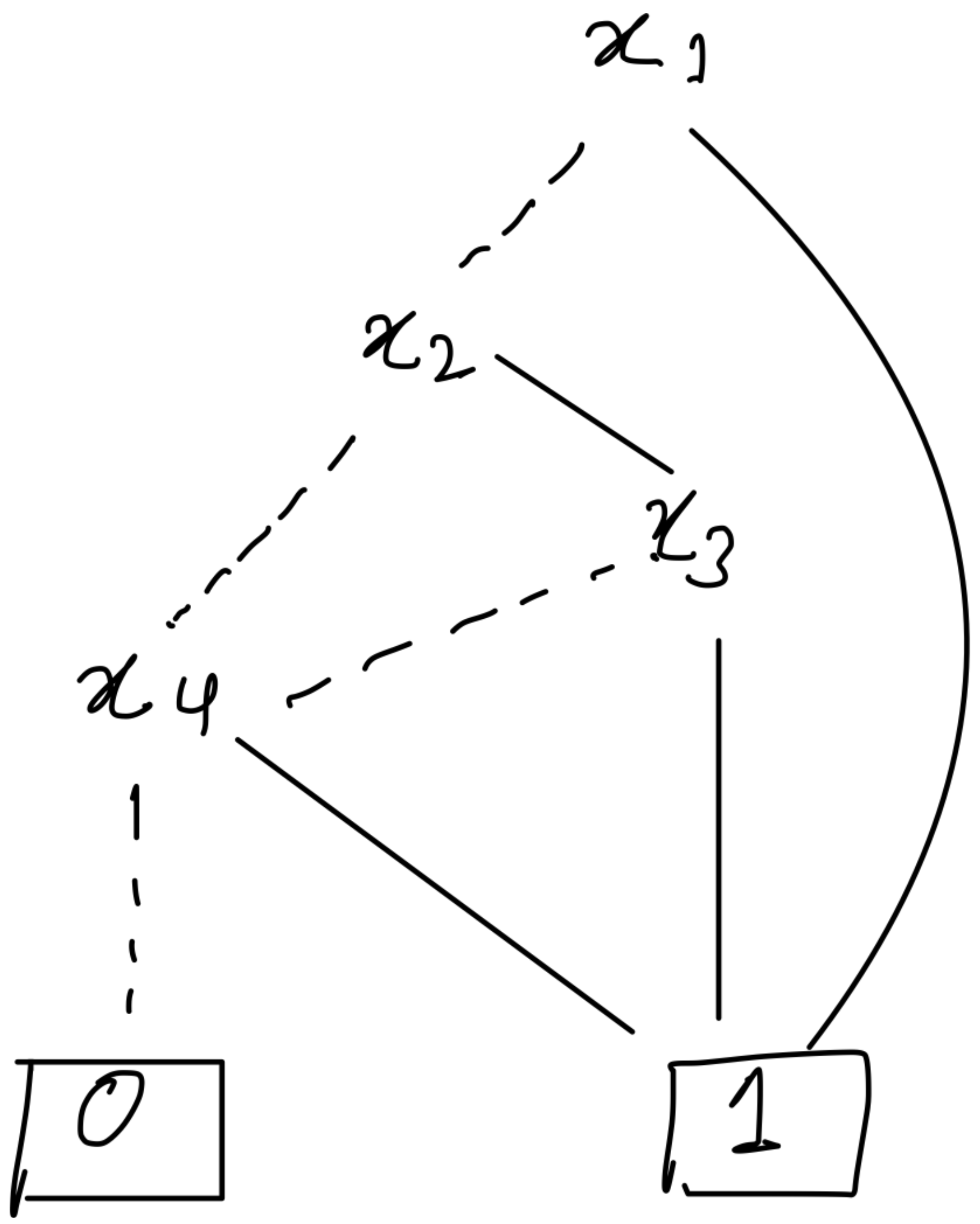


F



G



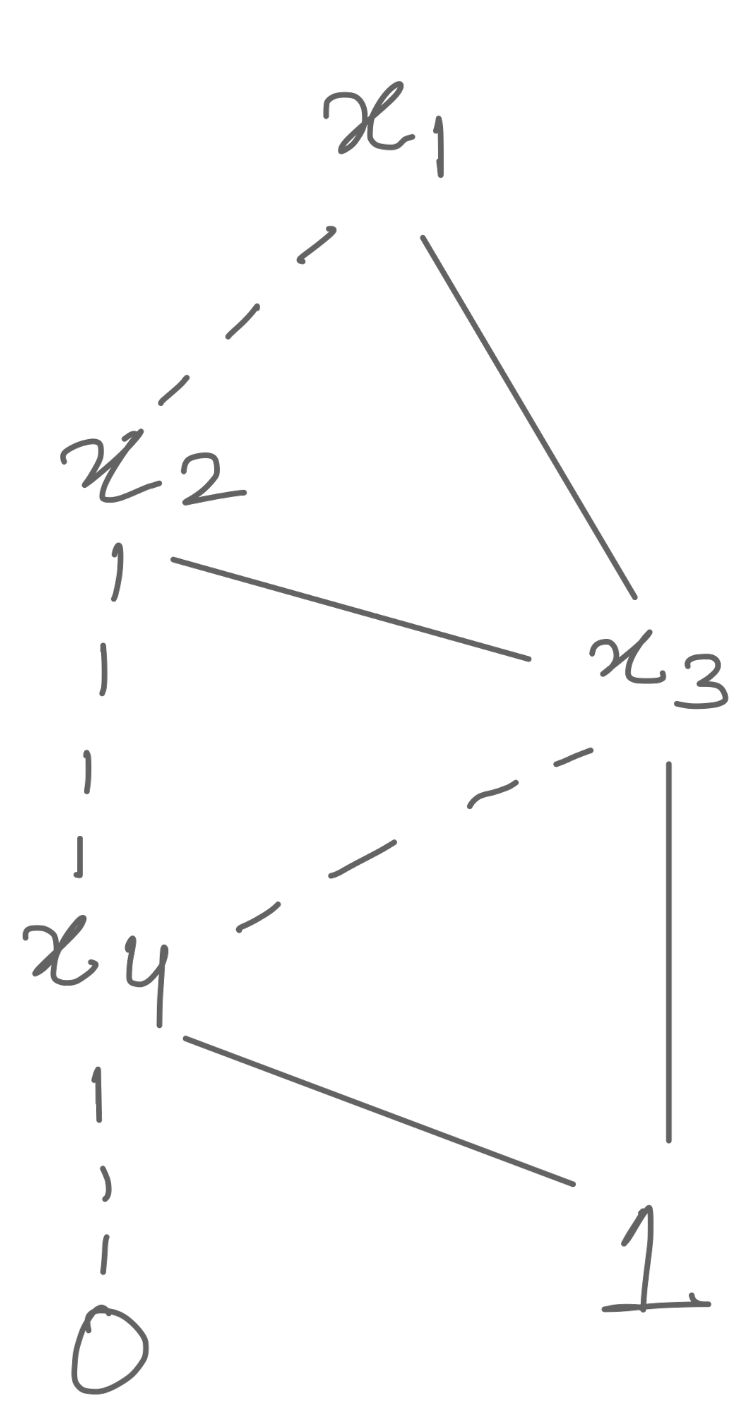
$F \vee G$

ROBDD Operations (Restrict)

Effect to setting a function argument x_i to a constant 0/1

Depth-first traversal.

Redirecting arcs according to constant.



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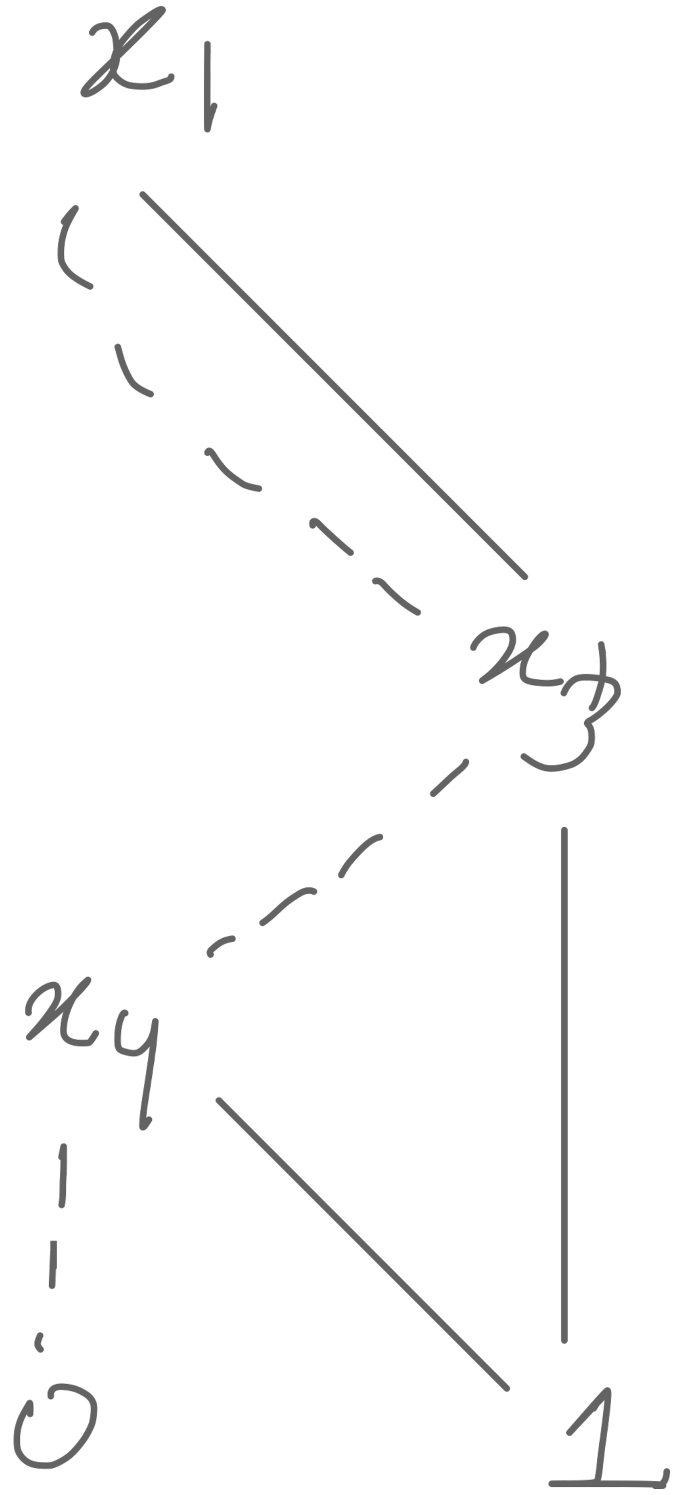
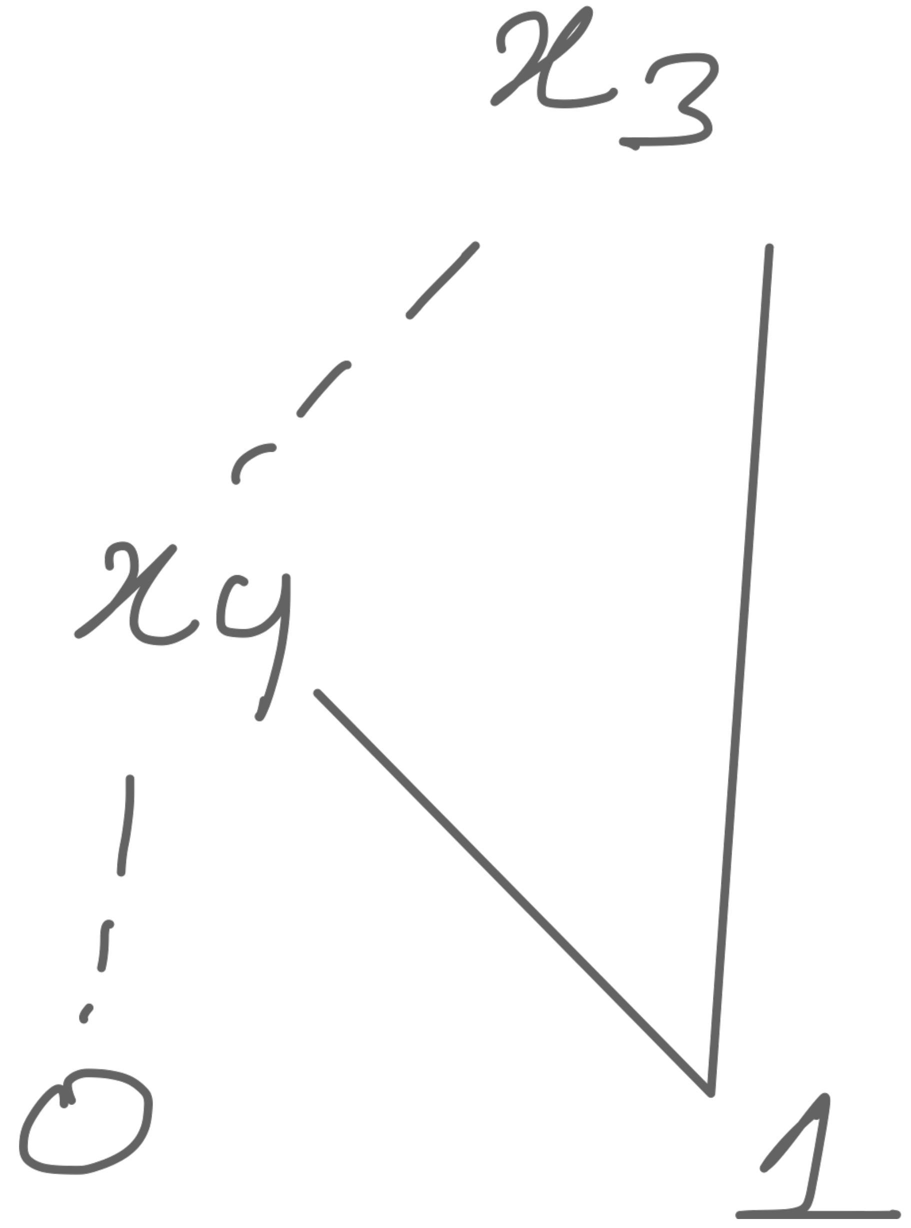


$F[x_2 = 1]$



$F[x_2 = 1]$

ROBDD Operations (Exists (x,F))

Compute ROBDD for $\exists x F$

1. Uses the identity:

$\exists x F \equiv F[x = 0] \vee F[x = 1]$

2. Realized using the restrict and apply functions

$$\text{Apply}(\vee, Restrict(x, 0, F), Restrict(x, 1, F))$$

$$\exists x_1, x_2 F \equiv ?$$

$$\exists x_1, x_2 F \equiv F(x_1, 1, x_3, \ldots, x_n) \vee F(x_1, 0, x_3, \ldots, x_n)$$

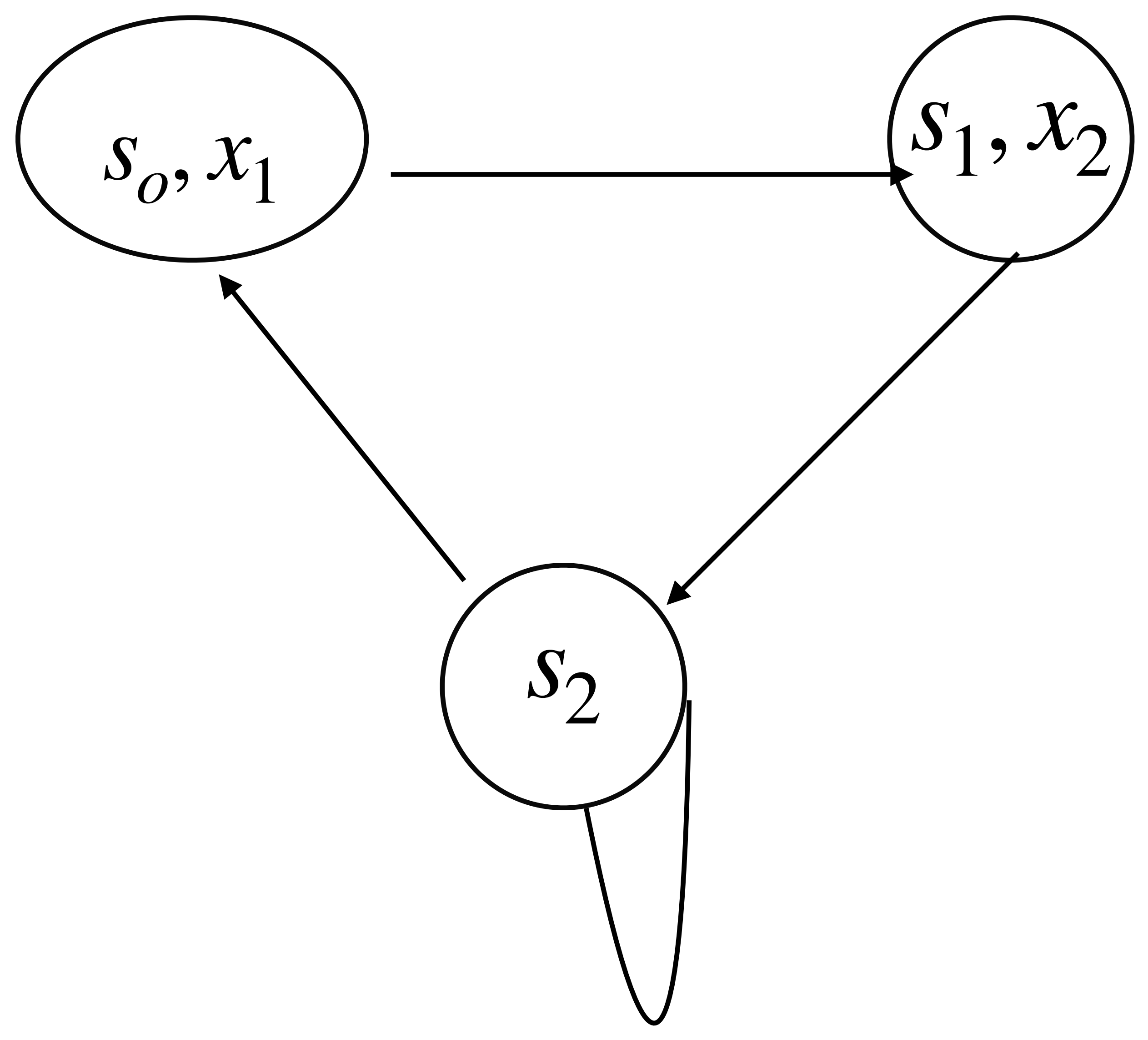
$$\exists x_1, x_2 F \equiv F(1, 1, x_3, \ldots, x_n) \vee F(0, 1, x_3, \ldots, x_n) \vee F(1, 0, x_3, \ldots, x_n) \vee F(0, 0, x_3, \ldots, x_n)$$

Implementing CTL Model Checking using BDDs

CTL model checking computes a set of states $[F_i]$ for every sub-formula F_i of the original formula F.

Sets of states will be represented using ROBDDs

That describes characteristic function of the set



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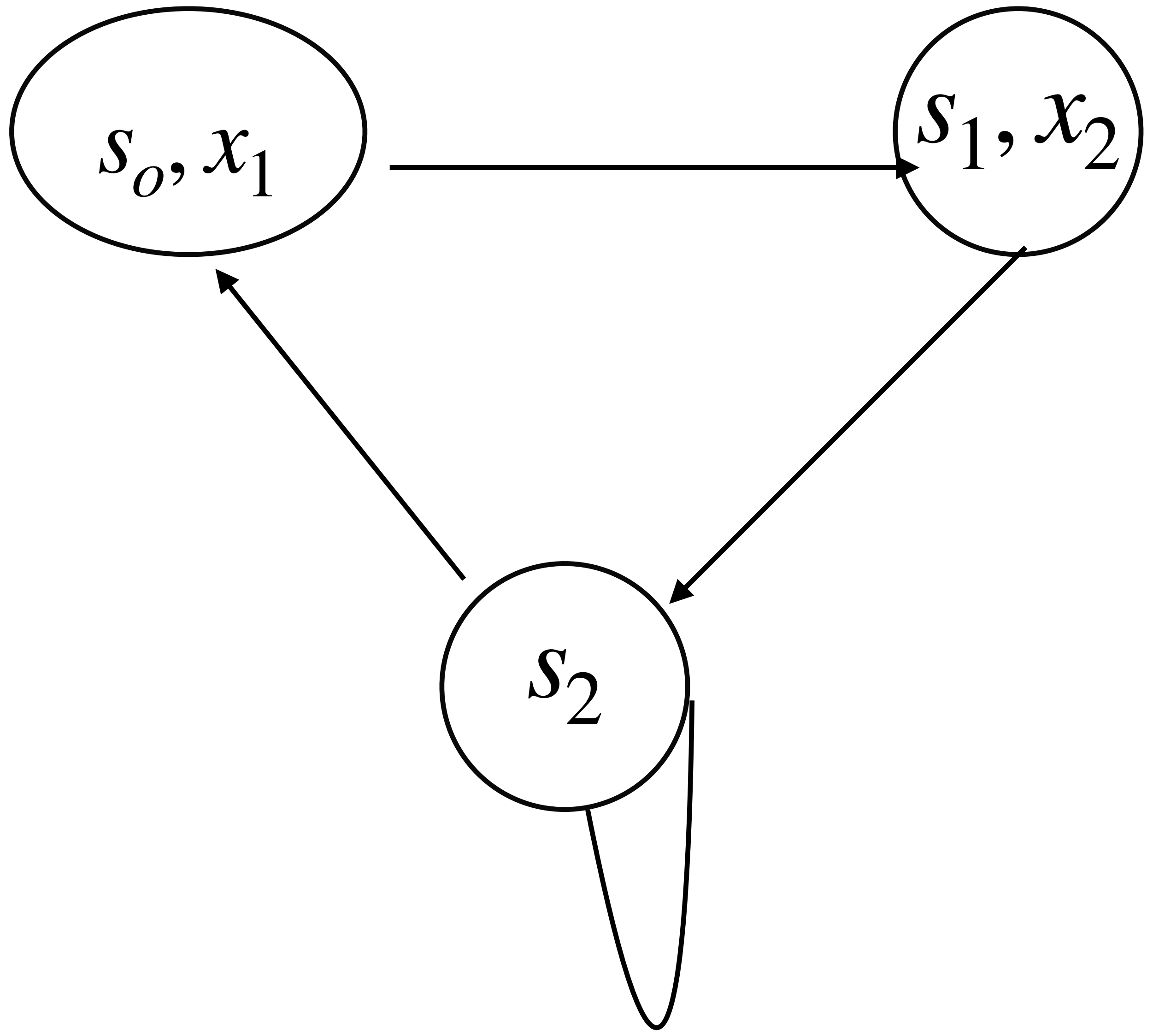

Set of states	Representation by	Representation by Boolean
∅		0
{so}	(1,0)	$x_1 . \neg x_2$
{s1}	(0,1)	$\neg x_1 . x_2$
{s2}	(0,0)	$\neg x_1 . \neg x_2$
{s0,s1}	(1,0),(0,1)	$x_1 . \neg x_2 + \neg x_1 . x_2$
{s0,s2}	(1,0),(0,0)	$x_1 . \neg x_2 + \neg x_1 . \neg x_2$
{s1,s2}	(0,1),(00)	$\neg x_1 . x_2 + \neg x_1 . \neg x_2$
{s0,s1,s2}	(1,0),(0,1),(0,0)	$x_1 . \neg x_2 + \neg x_1 . x_2 + \neg x_1 . \neg x_2$

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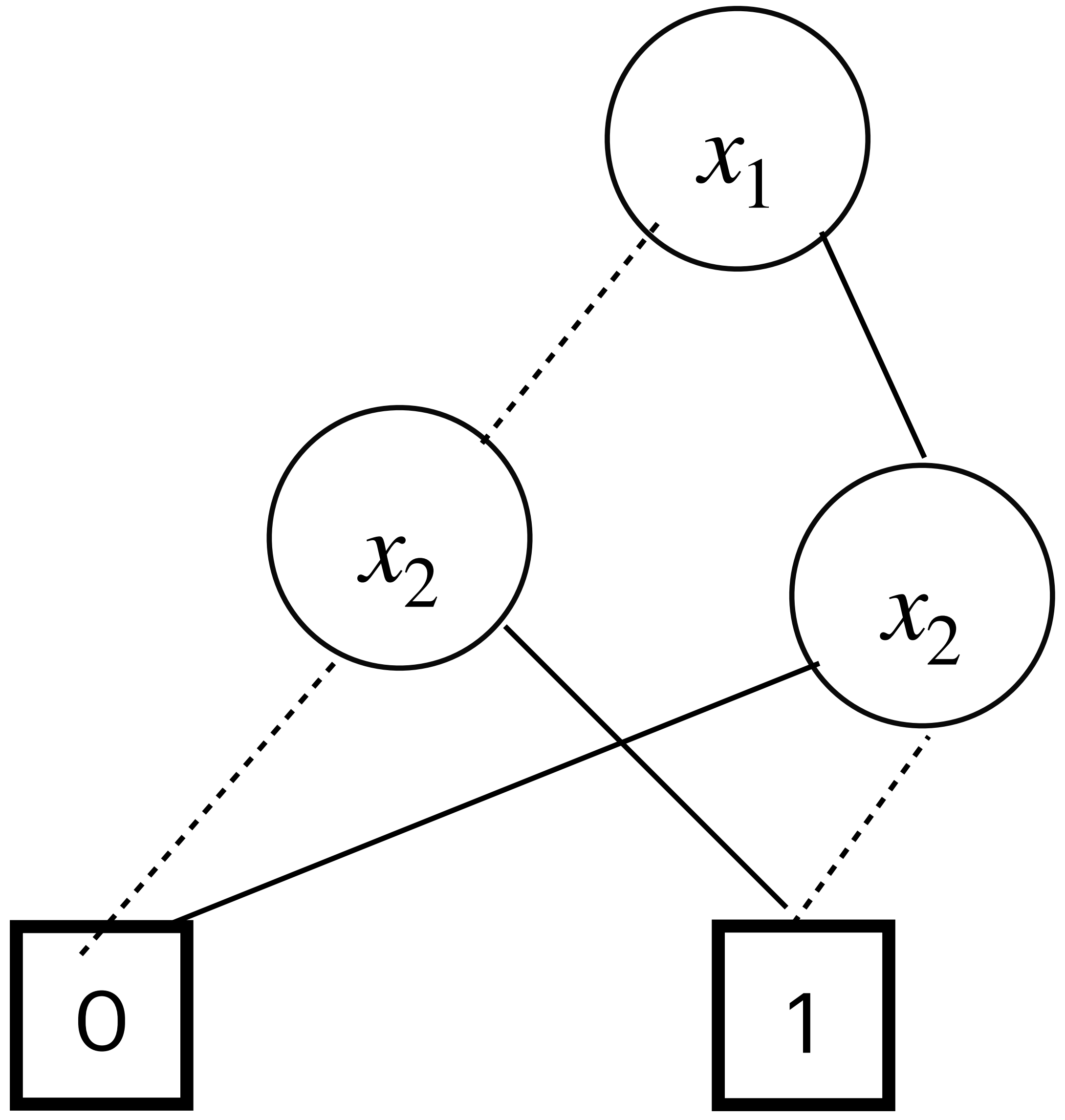

ROBDD for the set $\{s_o, s_1\}$

Set of states	Representation by	Representation by Boolean
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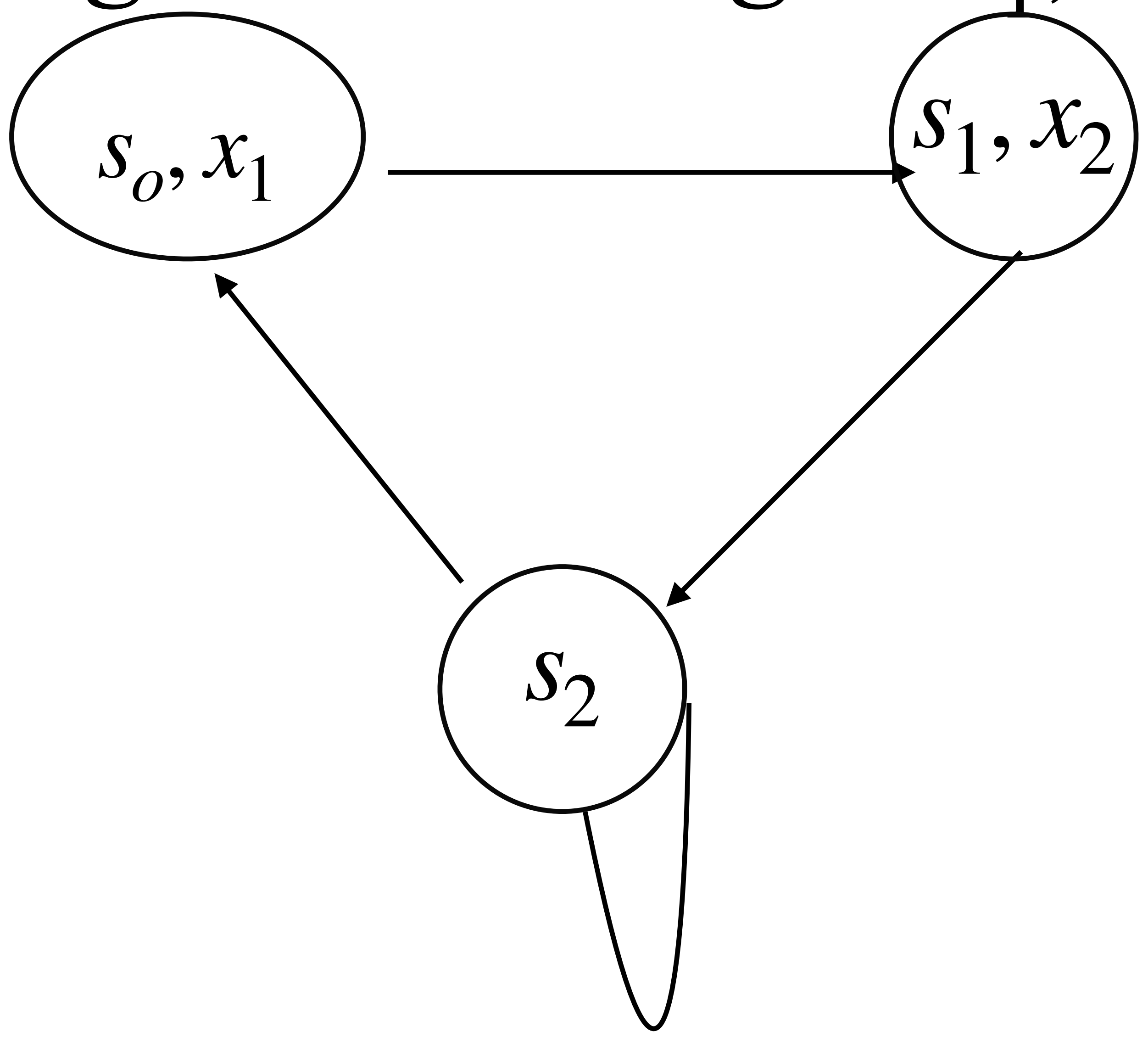
Implementing CTL Model Checking using BDDs

Representing the transition relations.

- Transition relations $(\rightarrow) \subseteq S \times S$ are represented by ROBDDs on 2n variables.
- If the variables $x_1, \ldots, x_n$ describe the current state, and the variables $x'_1, x'_2, \ldots x'_n$

describe the next state.

- The good ordering is $x_1, x'_1, x_2, x'_2, \ldots, x_n, x'_n$ (interleaving).



X1	X2	X'1	X'2	->
0	0	0	0	1
0	0	1	0	1
0	1	0	0	1
1	0	0	1	1
0	0	0	1	0
..	..	..	..	..

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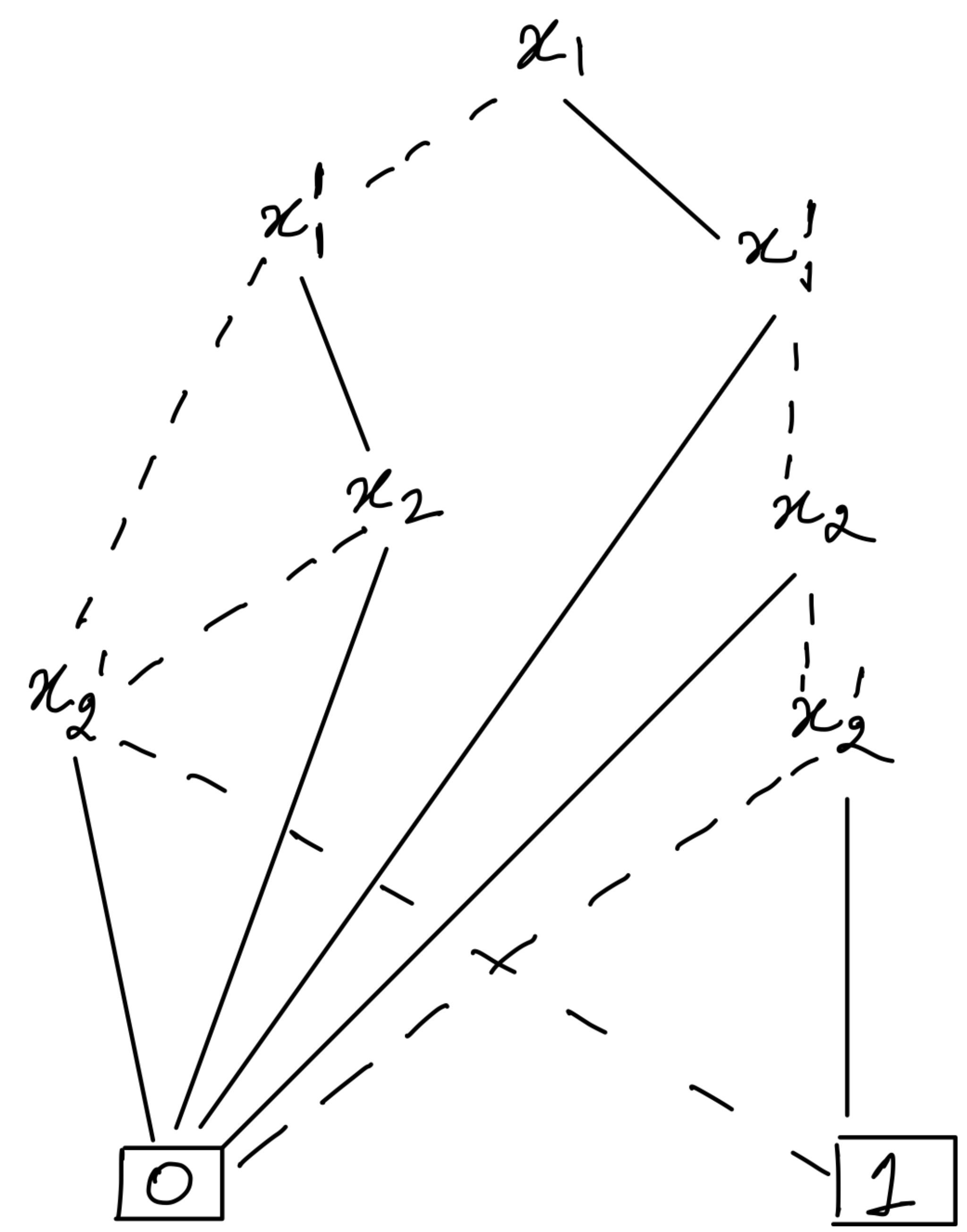
ROBDD of $F^{\rightarrow}$



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describe the next state. The good ordering is $x_1, x'_1, x_2, x'_2, \ldots, x_n, x'_n$ (interleaving).

But exploring Truth table will be expensive.

Can we learn $F^{\rightarrow}$ without Truth table?

X1	X2	X'1	X'2	->
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0	0	1	0	1
0	1	0	0	1
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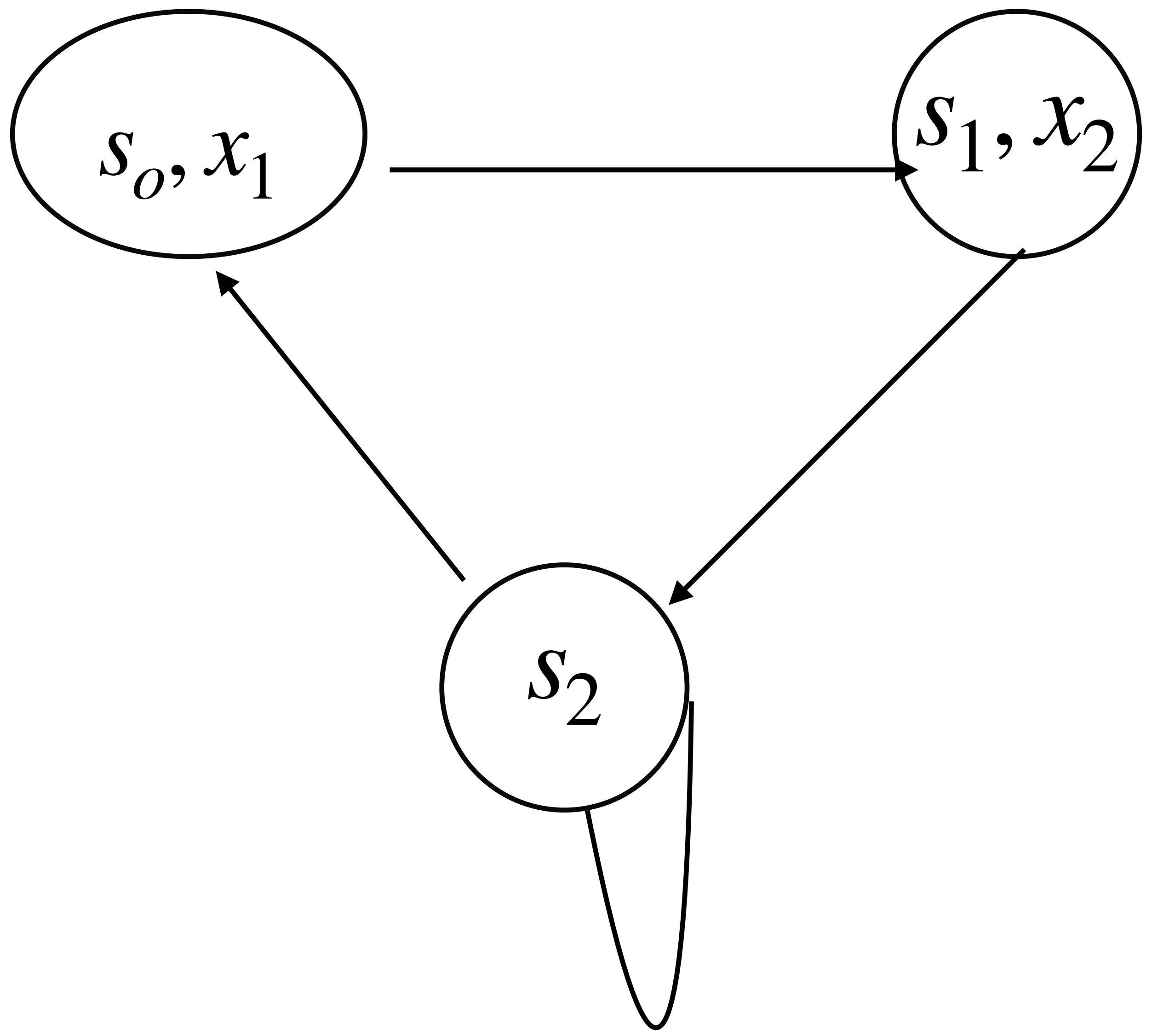
Can we learn $F^{\rightarrow}$ without Truth table?



$$F^{\rightarrow} := (x_1 \wedge \neg x_2 \wedge \neg x'_1 \wedge x'_2) \vee (\neg x_1 \wedge x_2 \wedge \neg x'_1 \wedge \neg x'_2) \vee (\neg x_1 \wedge \neg x_2 \wedge \neg x'_1 \wedge \neg x'_2) \vee (\neg x_1 \wedge \neg x_2 \wedge x'_1 \wedge \neg x'_2)$$

Convert $F^{\rightarrow}$ to ROBDD.

Implementing CTL Model Checking using BDDs

Symbolic Model Checking — it represents and manipulates sets of states and transitions using symbolic expressions or formulas (like Boolean functions or Binary Decision Diagrams) rather than explicitly enumerating each state.

Specification — $F = \exists \mathbf{N} p$

Pre([p]) same as Pre(Y)

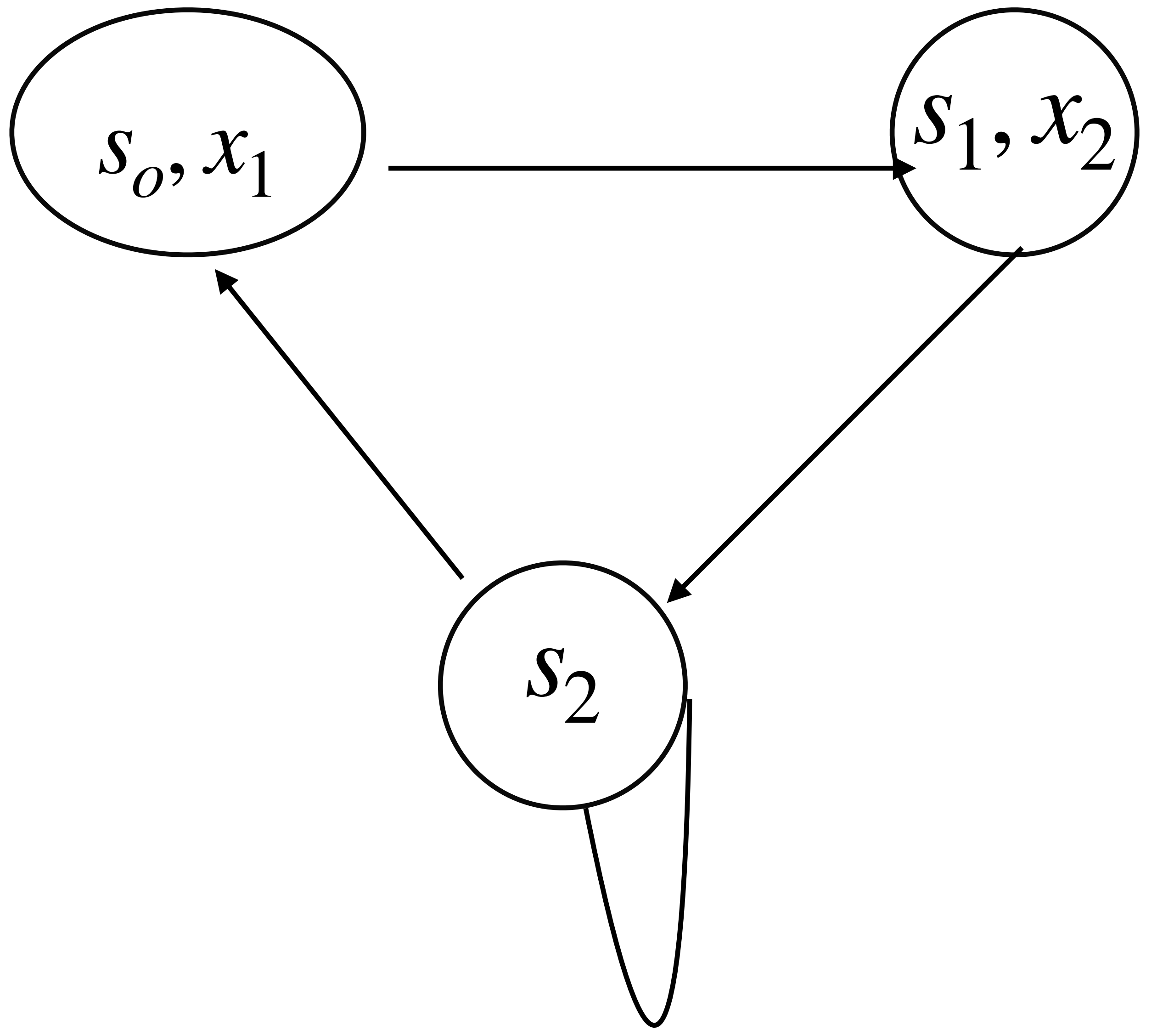
$$B_{Pre(Y)} = \text{ exists } (X', \text{apply}(\wedge, F^{\rightarrow}, F_{Y'}))$$

Where X' is set of next state variables.

$F^{\rightarrow}$ is the ROBDD representing the transition relation.

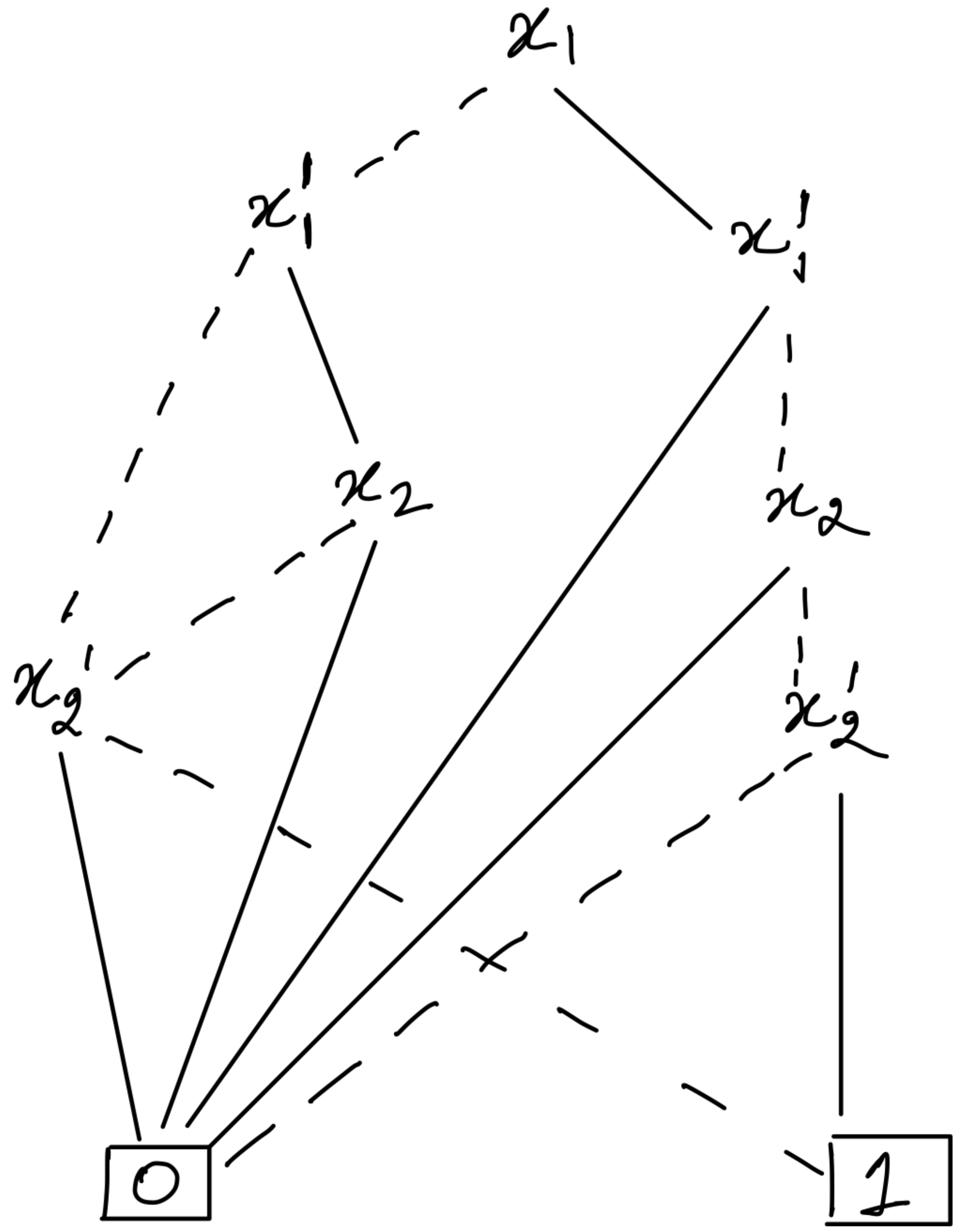
$F_{Y'}$ is the ROBDD representing the set Y with variables $x_1, x_2, \ldots, x_n$ renamed to $x_1', x_2', \ldots, x_n'$

Symbolic Model Checking



$S = x_1 . \neg x_2 + \neg x_1 . x_2 + \neg x_1 \neg x_2$

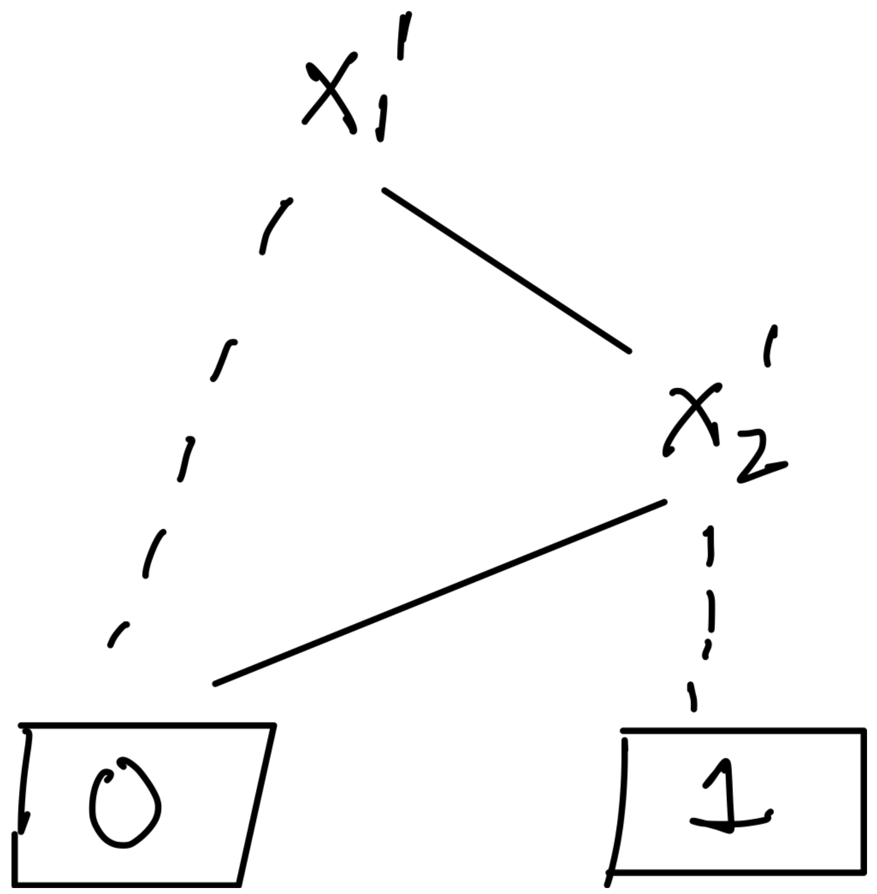
ROBDD of $F^{\rightarrow}$



$\exists \mathbf{N} x_1$

$$B_{Pre(Y)} = \text{ exists } (X', \text{apply}(\wedge , F^{\rightarrow}, F_{Y'}))$$

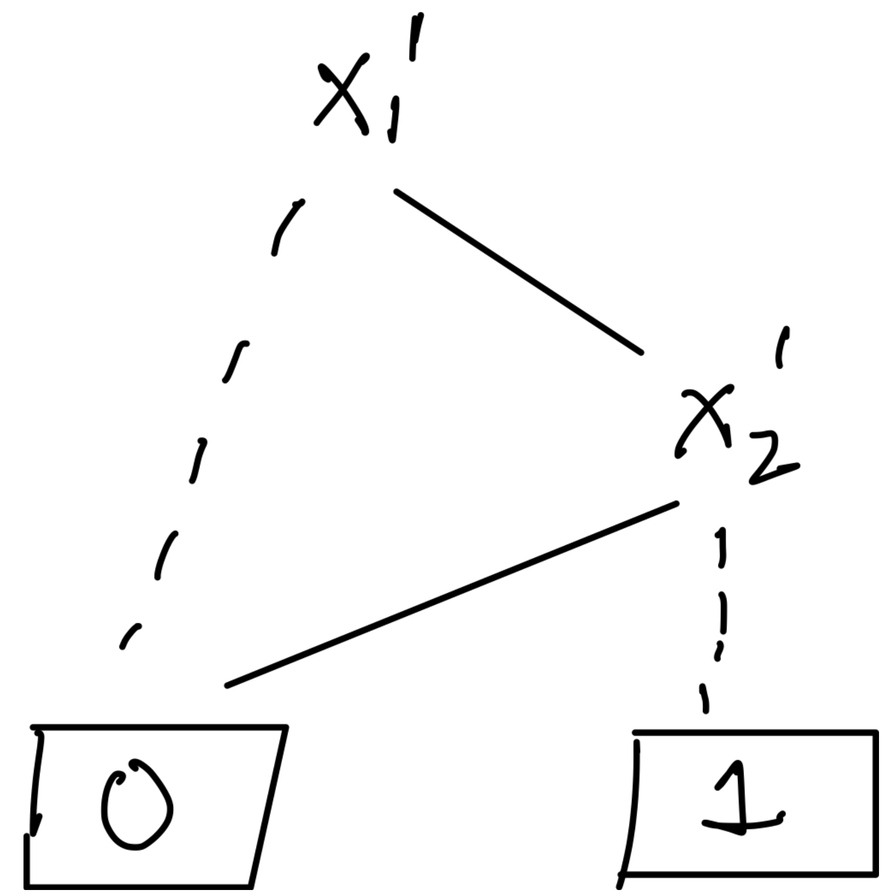
$$F_{Y'} = ROBDD(s_0)$$



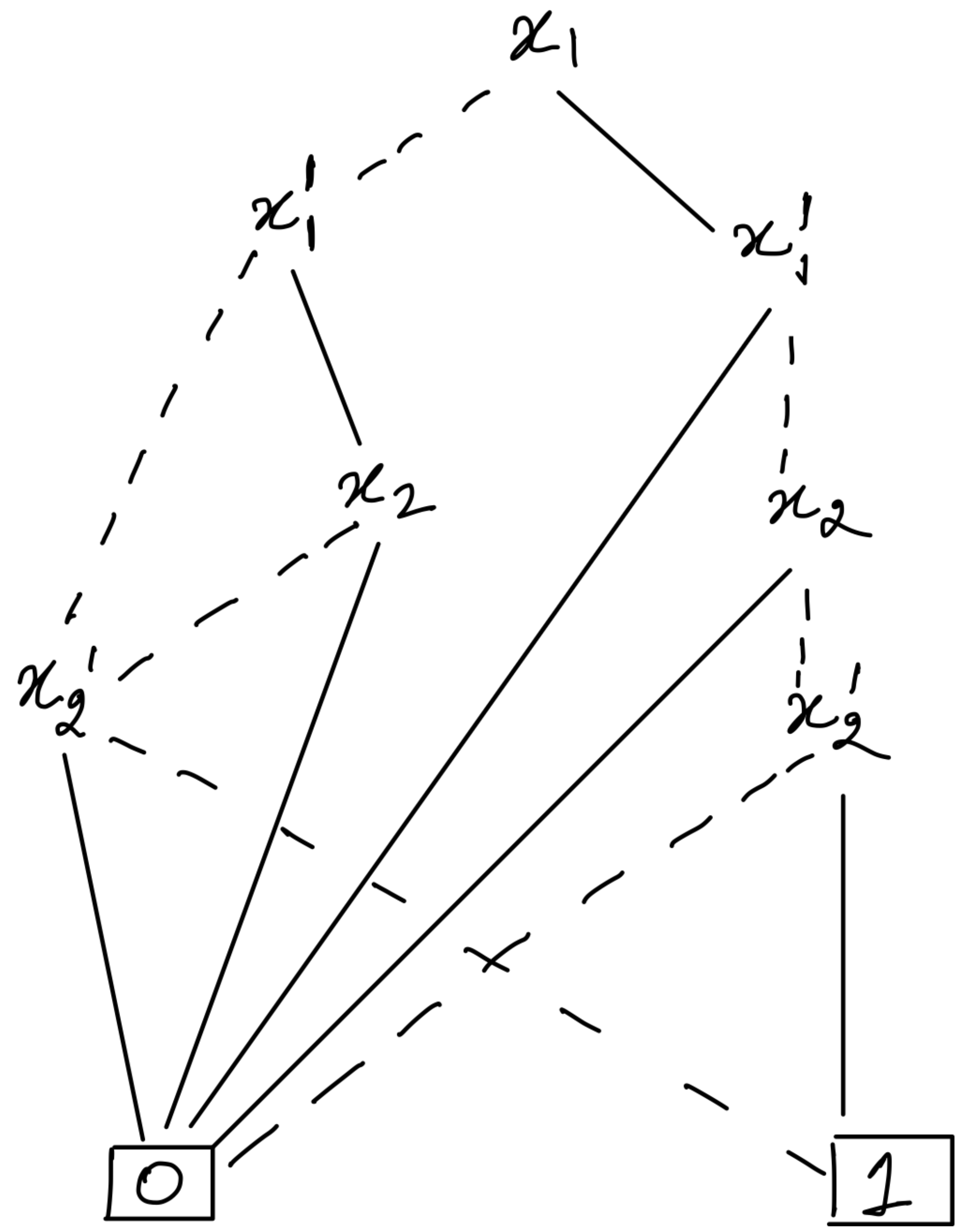
Symbolic Model Checking

$B_{Pre(Y)} = \text{ exists } (X', \text{apply}(\wedge, F^{\rightarrow}, F_{Y'}))$

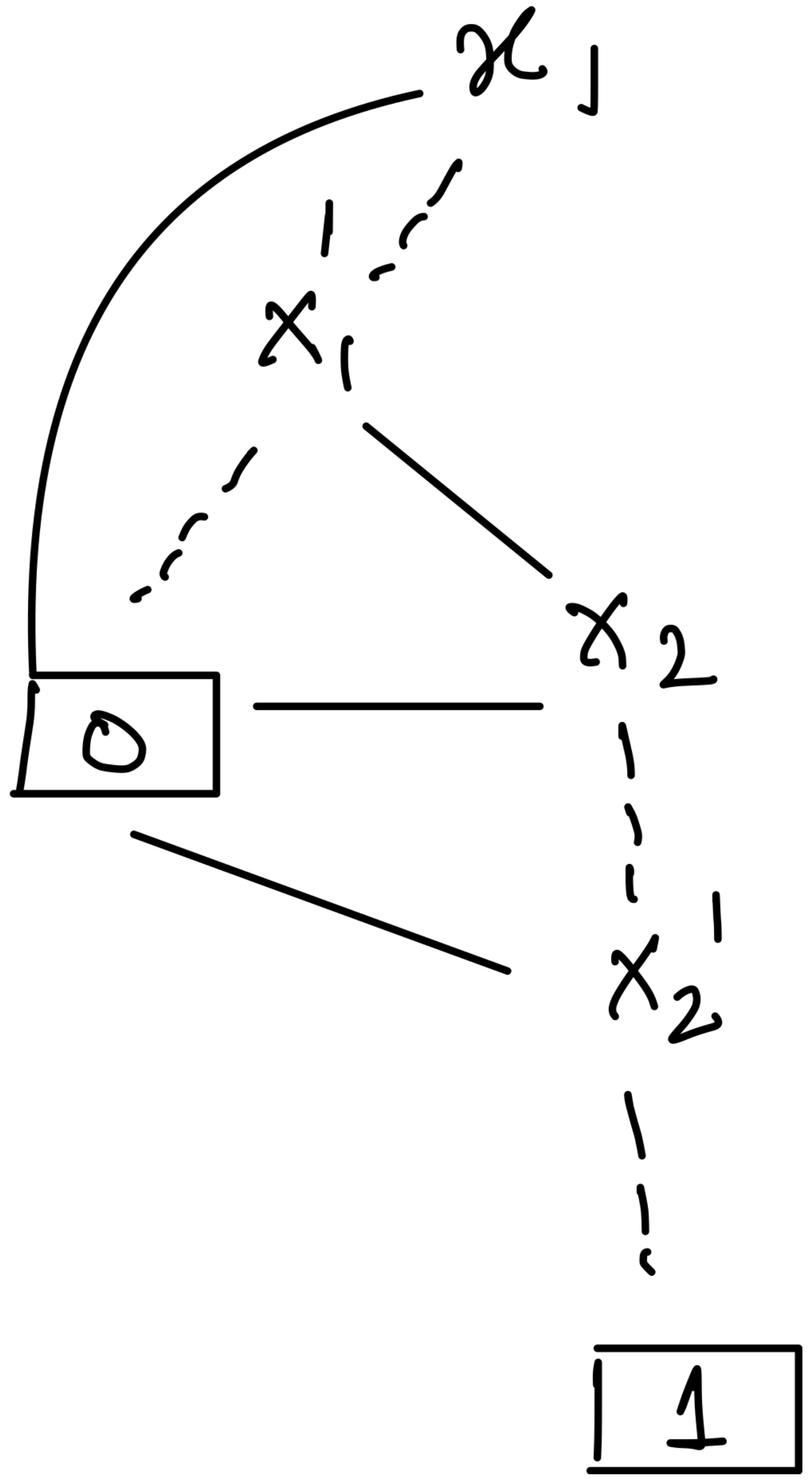
$B_{Pre(Y)} = exists(x_1', x_2', apply(\wedge, F^{\rightarrow}, F_{Y'})$



$F_{Y'} = ROBDD(s_0)$



ROBDD of $F^{\rightarrow}$



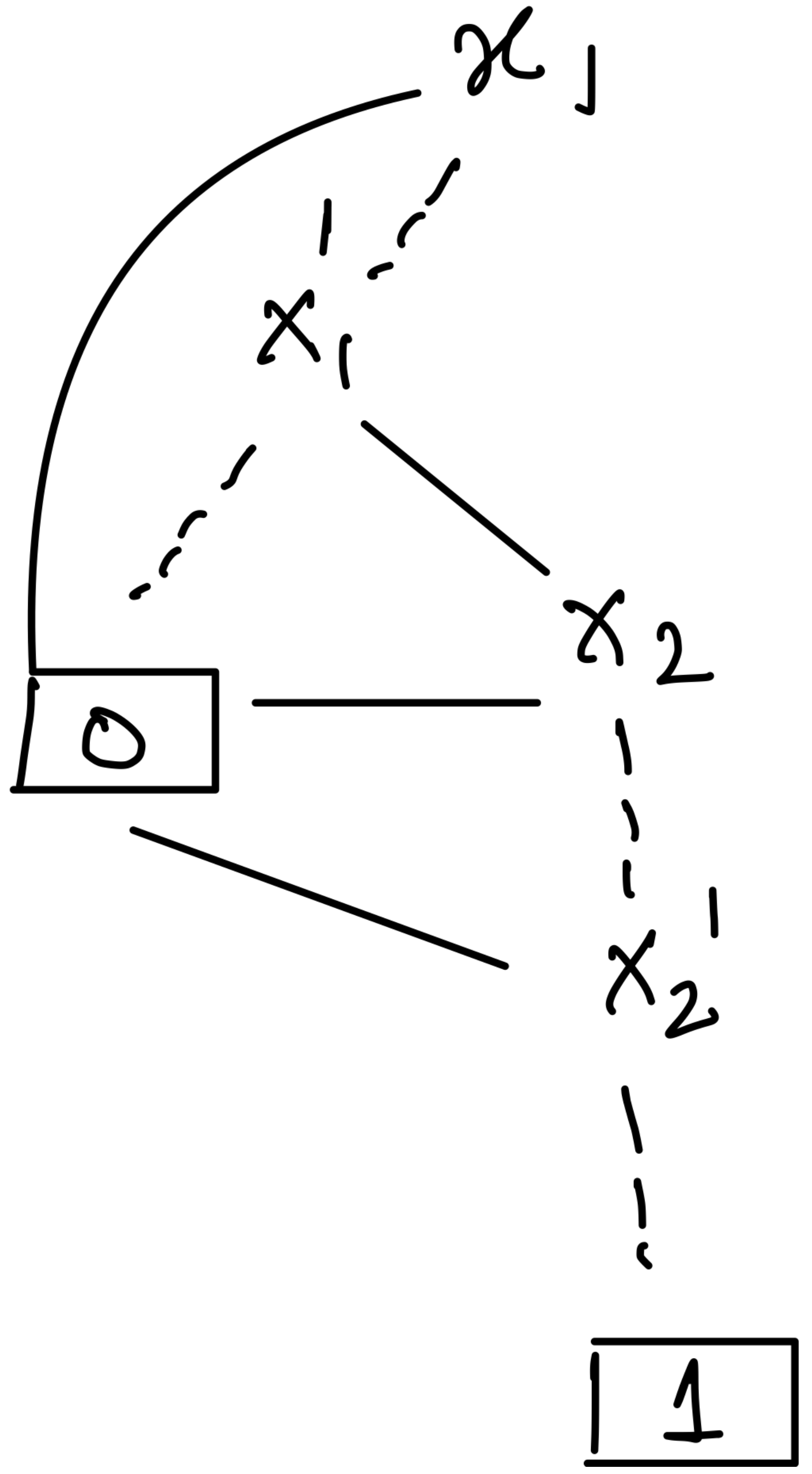
$apply(\wedge, F^{\rightarrow}, F_{Y'})$

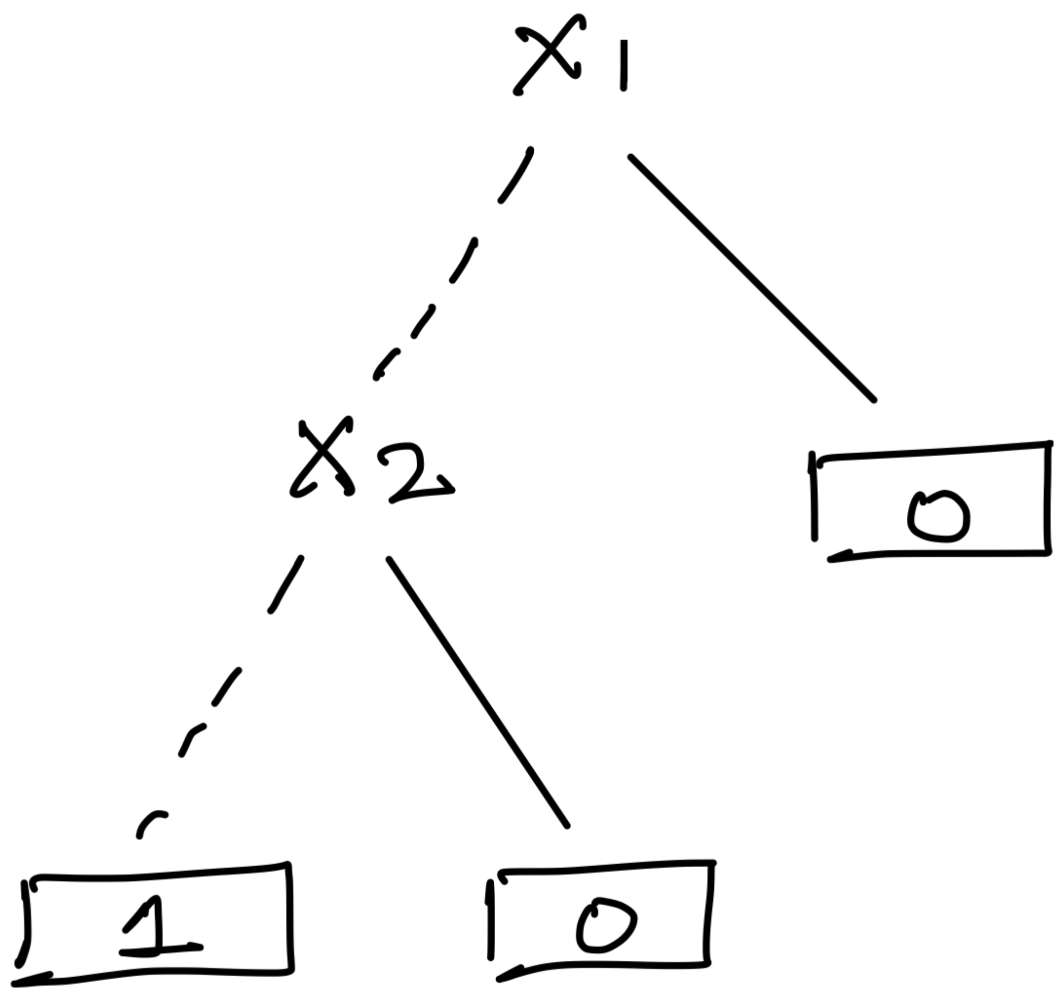
Symbolic Model Checking

$$B_{Pre(Y)} = exists(x'_1, x'_2, apply(\wedge, F^{\rightarrow}, F_{Y'}))$$

$$B_{Pre(Y)} =$$

$$restrict(x_1, x_2, F_1, 0,0) \vee restrict(x_1, x_2, F_1, 1,0) \vee restrict(x_1, x_2, F_1, 0,1) \vee restrict(x_1, x_2, F_1, 1,1)$$





$$restrict(x_1, x_2, F_1, 0,0) \vee restrict(x_1, x_2, F_1, 0,1) =$$

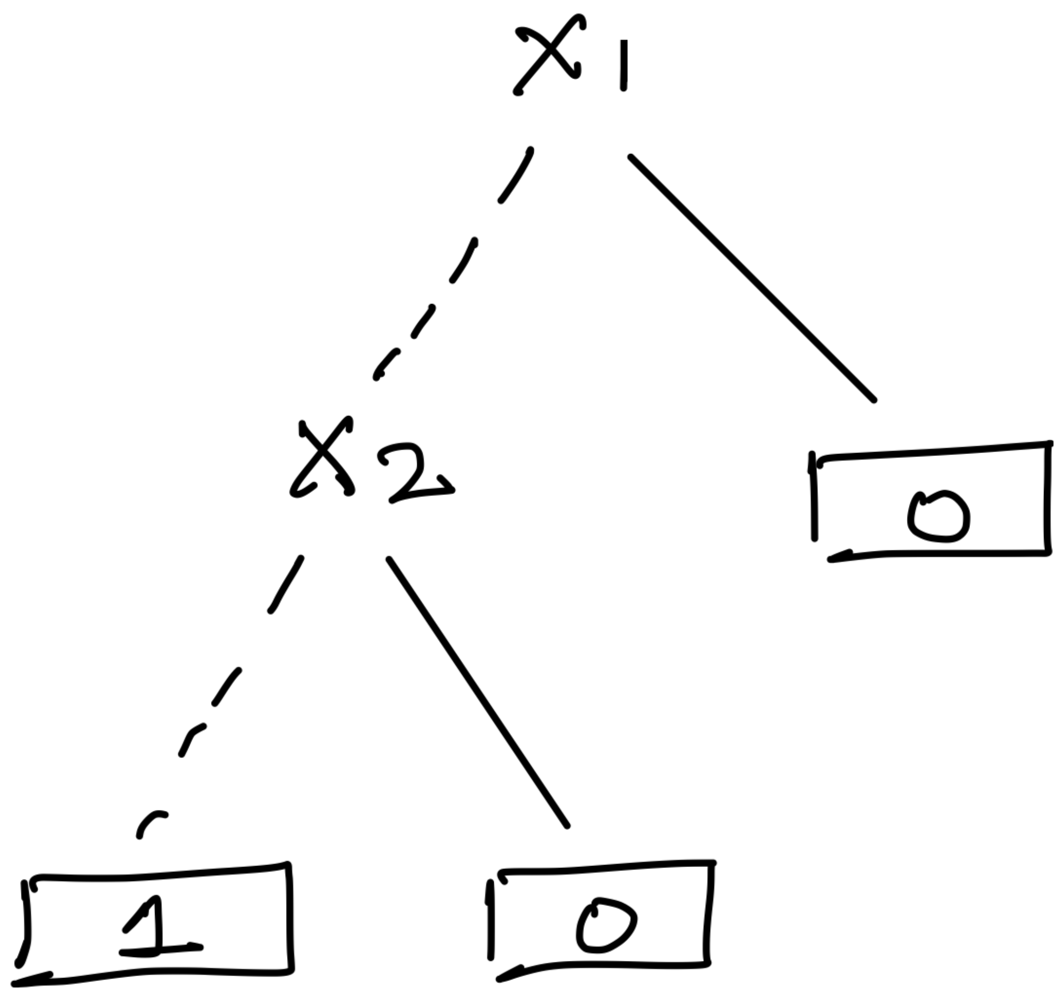
$$restrict(x_1, x_2, F_1, 0,1) = 0$$

$$restrict(x_1, x_2, F_1, 1,0)$$

$$F_1 = apply(\wedge, F^{\rightarrow}, F_{Y'})$$

Symbolic Model Checking

$$B_{Pre(Y)} = exists(x_1', x_2', apply(\wedge, F^{\rightarrow}, F_{Y'})$$



ROBDD of s_2

CTL Model Checking Algorithm —Symbolic Model Checking

Function Label$(F, M)\{$

Case F of :

True	return S
False	return {}
p	return $\{s \in S \mid p \in L(s)\}$
$\neg F_1$	return $\neg ROBDD\ of\ F_1$
$F_1 \wedge F_2$	return $apply(\wedge, ROBDD(F_1), ROBDD(F_2))$
$\exists \mathbf{N} F_1$	return $\mathrm{pre}(ROBDD(F_1'), ROBDD(F^{\rightarrow}))$
$\exists \Box F_1$	return $Label\_EG(ROBDD(F_1'), ROBDD(F^{\rightarrow}))$
$\exists F_1 \mathbf{U} F_2$	return $Label\_EU(ROBDD(F_1'), ROBDD(F_2'), ROBDD(F^{\rightarrow}))$

End Case