Automated Synthesis: Towards the Holy Grail of AI

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AAAI-2022

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that could work like
this ...

What’s It All About

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Spec by examples

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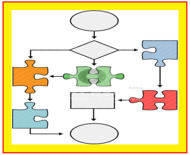

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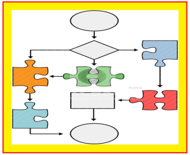

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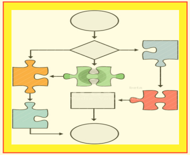

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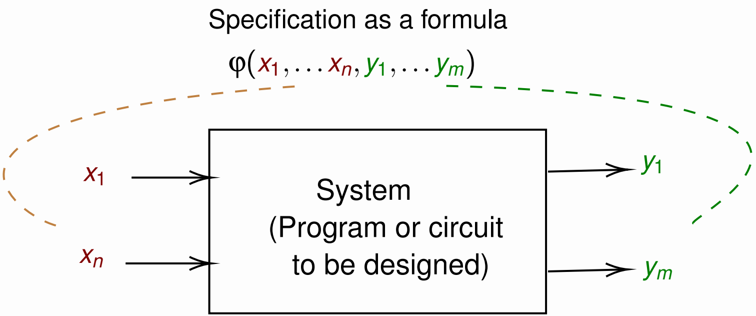
Automated Synthesis: A Generic View



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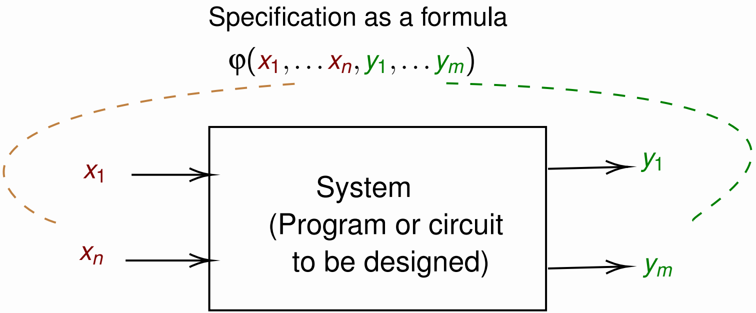

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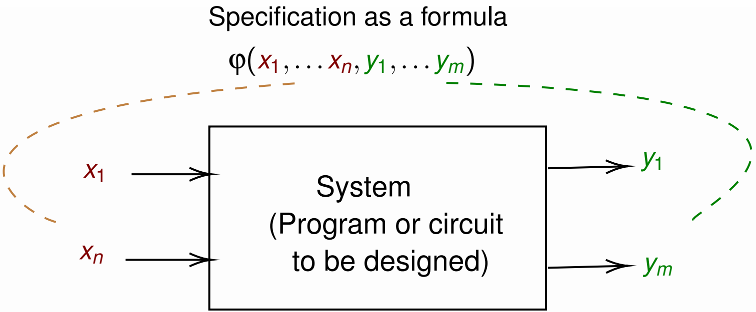

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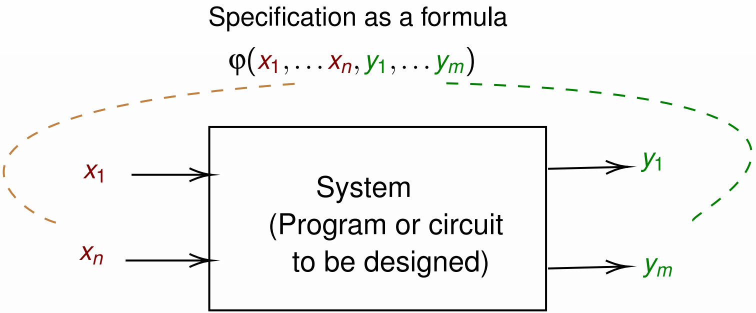

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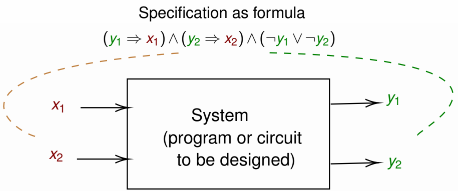
  - x_i *input* variables (vector **X**)
  - y_j *output* variables (vector **Y**)
- Need **Y** as functions **F** of
  - "History" of **X** and **Y**, "State" of system, ... in general

such that $\varphi(\mathbf{X}, \mathbf{F})$ is satisfied.

Automated Synthesis: Concrete View 1 (Memoryless Arbiter)



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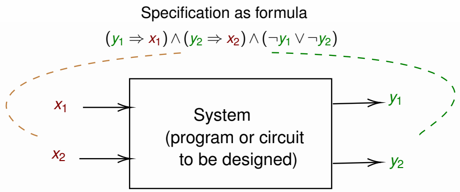

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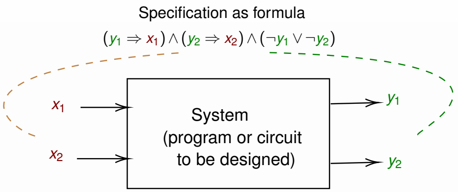

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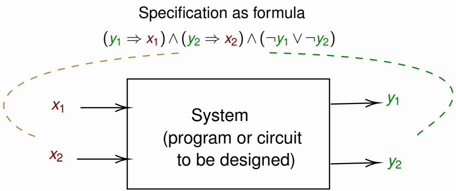

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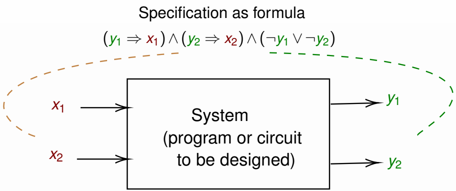

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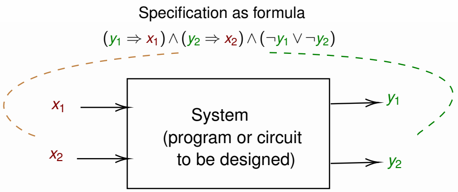

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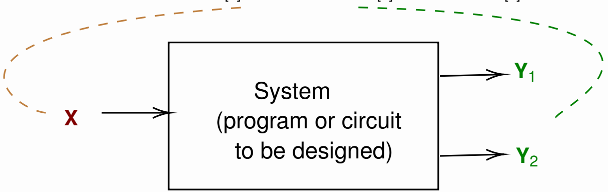
- Suffices to give one "good enough" solution

Automated Synthesis: Concrete View 2 (Cryptanalysis)

Specification as bit-vector formula

$$(\mathbf{X} = \mathbf{Y}_1 \times_{[n]} \mathbf{Y}_2) \wedge \neg(\mathbf{Y}_1 = \mathbf{1}_{[n]}) \wedge \neg(\mathbf{Y}_2 = \mathbf{1}_{[n]})$$



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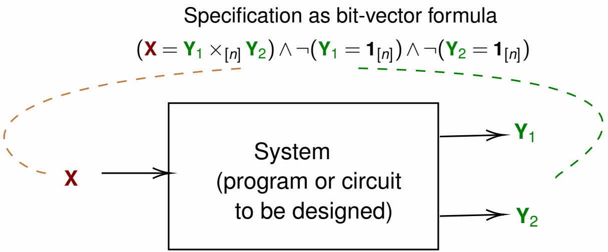

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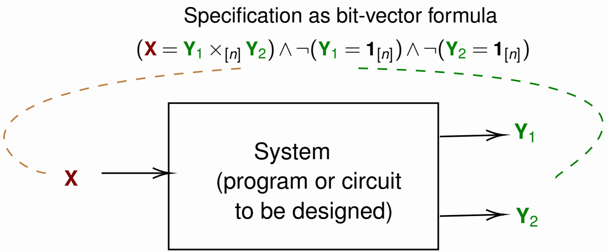

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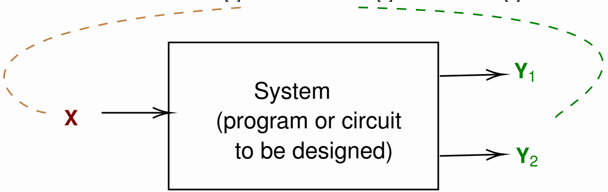

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  - Efficient solution would break crypto systems

Reactive vs Functional Synthesis

Reactive synthesis

- System & environment in continuous temporal interaction
- Specification talks of **infinite sequence of inputs & outputs**
  - Temporal logic, automata over infinite words, ...
- Examples: Operating system, network switch, nuclear plant controller, ...

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2. Application domains
3. Theoretical hardness and practical algorithms

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A coffee/tea/dinner/drinks break

Second half: Under the hood

4. Deep Dives
    1. Knowledge compilation
    2. Counter-example guided
    3. Data-driven approaches
5. Tool demos and experimental results
6. Conclusion and the way forward

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Boolean Functional Synthesis

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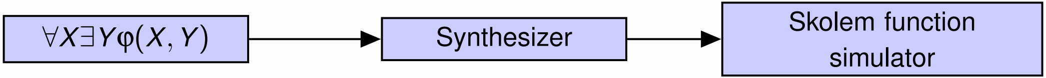
  - Given $\mathbf{X}$, $\mathbf{F}(\mathbf{X})$, easy to check if $\exists \mathbf{Y}\ \varphi(\mathbf{X}, \mathbf{Y}) = \varphi(\mathbf{X}, \mathbf{F}(\mathbf{X})) = 0$

A short tool demo



In a specific format

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Specifications

- Logical specifications
- Test cases (examples)
- Natural Language
- Demonstrations/Traces
- Programs

Application Domain 1: Syntax-Guided Synthesis (SyGuS) formalization*

SyGuS was an attempt to formalize the core synthesis problem as a:

- a background theory (eg. QF_UFLIA)
- a semantic correctness specification (in the background theory)
- a language to represent the synthesized program (as a context-free grammar)

* Alur et al.,FMCAD'13

Application Domain 1: Developer Assistance: Heap Manipulations*

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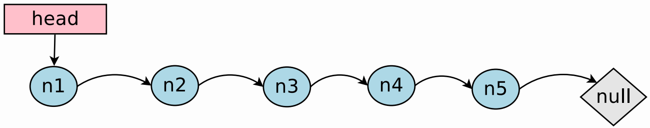

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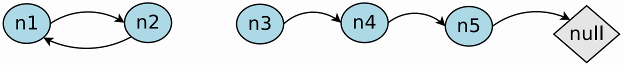

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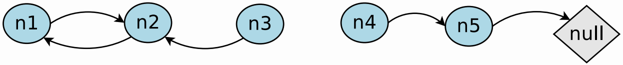

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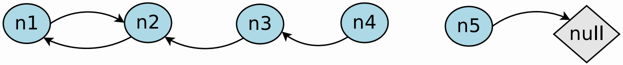

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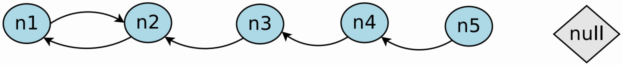

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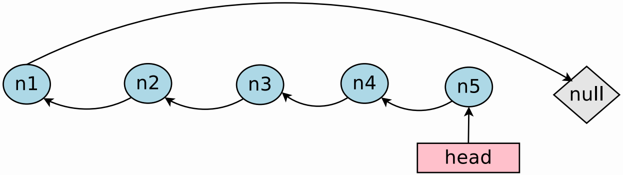

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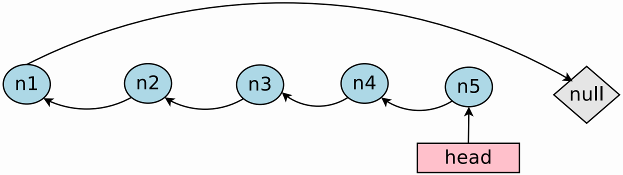

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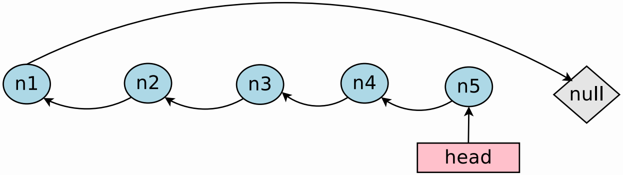


```
0 tmp1 = x
1 tmp2 = tmp1.next
  while(not (tmp2 == null))
2   tmp0 = tmp2.next
3   tmp2.next = x
4   x = tmp2
5   tmp2 = tmp0
6 tmp1.next = tmp0
```

*Roy, SAS'13; Garg and Roy, SAS'15; Verma and Roy, ESEC/FSE'17; Verma and Roy, FMSD'22

- **Data preparation**: synthesize R scripts for complex data wrangling tasks*

---

*Feng et al., PLDI’17

†Wang et al., PLDI’17; https://scythe.cs.washington.edu/

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- **Data preparation**: synthesize R scripts for complex data wrangling tasks*
- **Data extraction**: synthesize SQL queries† and Python scripts‡ from examples of input and output tables

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Application Domain 1: Synthesis for Data Scientists

- **Data preparation**: synthesize R scripts for complex data wrangling tasks[*]
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- **Data visualization**: automatically synthesize visualizations from data with small example(s) as input [§]

[*] Feng et al., PLDI'17

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- **Data visualization**: automatically synthesize visualizations from data with small example(s) as input §
- **ML pipelines**: allows for generating supervised learning pipelines¶

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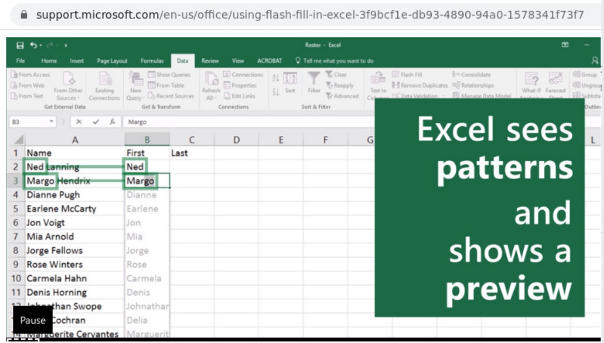
Flash Fill (Microsoft Excel)*

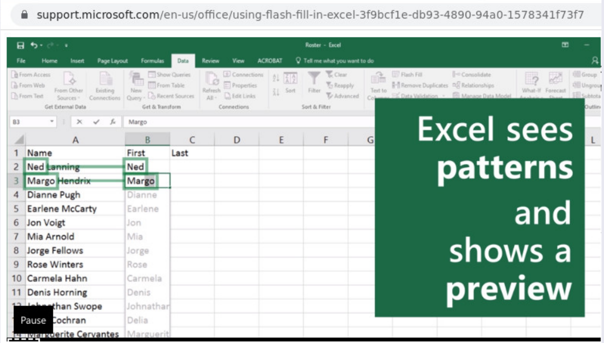


- automatically identifies patterns, and
- synthesizes a program in the background

*Gulwani et al., POPL'11; image and video at https://support.microsoft.com

†Singh and Gulwani, PVLDB'12; Singh and Gulwani, CAV'12; Harris and Gulwani, PLDI'11

Flash Fill (Microsoft Excel)[*]



- automatically identifies patterns, and
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Similar line of tools for semantic string, number and table transformations.[†]

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Application Domain 1: Intelligent Tutoring

- **Problem Generation**: for geometry, natural deduction and arithmetic*

---

*Alvinet al., AAAI'14; Ahmedet al., IJCAI'13; Andersen et al.,CHI'13

†Gulwani et al., PLDI'11

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- **Problem Generation**: for geometry, natural deduction and arithmetic*
- **Solution Generation**: geometry constructions†

$MidPoint(Line(p_1, p_2))$

$c_1 = Circle(p_1, len(Line(p_1, p_2)))$
$c_2 = Circle(p_2, len(Line(p_1, p_2)))$
$q_1, q_2 = CircleCircleXn(c_1, c_2)$
$r = LineLineXn(Line(p_1, p_2), Line(q_1, q_2))$
return r
(simplified for presentation)

*Alvinet al., AAAI'14; Ahmedet al., IJCAI'13; Andersen et al.,CHI'13

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(simplified for presentation)

- **Feedback Generation**: introductory programming and automata‡

---

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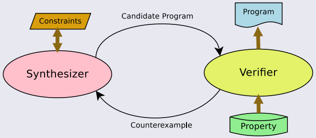
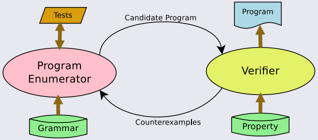
Application Domain 1: Algorithms for Program Synthesis*

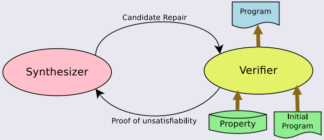
CEGIS (Symbolic)



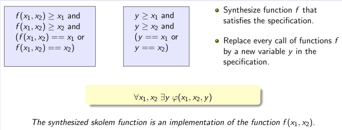
CEGIS (Enumerative)



SyPR: Proof-Guided Repairs



Reduction to QBF



*CEGIS(Sym): Solar-Lezama, STTT'12. CEGIS(Enum): Alur et al., FMCAD'13; Alur et al., TACAS'17; SyPR: Verma and Roy, ESEC/FSE'17; Verma et al., CGO'20; Golia et al., CAV'20; RedQBF: Golia et al., IJCAI'21

Application Domain 1: Program synthesis to Skolem Functional Synthesis*

$$f(x_1, x_2) \geq x_1 \text{ and}$$
$$f(x_1, x_2) \geq x_2 \text{ and}$$
$$(f(x_1, x_2) == x_1 \text{ or}$$
$$f(x_1, x_2) == x_2)$$

- Synthesize function f that satisfies the specification.

*Golia et al., IJCAI'21

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$f(x_1, x_2) \geq x_1$ and
$f(x_1, x_2) \geq x_2$ and
$(f(x_1, x_2) == x_1$ or
$f(x_1, x_2) == x_2)$

$y \geq x_1$ and
$y \geq x_2$ and
$(y == x_1$ or
$y == x_2)$

- Synthesize function f that satisfies the specification.
- Replace every call of functions f by a new variable y in the specification.

$$\forall x_1, x_2 \, \exists y \; \varphi(x_1, x_2, y)$$

*Golia et al., IJCAI'21

Application Domain 1: Program synthesis to Skolem Functional Synthesis*

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$$\forall x_1, x_2 \, \exists y \; \varphi(x_1, x_2, y)$$

The synthesized skolem function is an implementation of the function $f(x_1, x_2)$.

*Golia et al., IJCAI'21

Application Domain 2: Games and planning

Conway's Game of Life

- Infinite 2D grid of cells, each alive or dead in each gen:
  1. (Under-pop) live cell with <2 live neighbors dies;
  2. (Status-quo) live cell with 2 or 3 live neighbors lives;
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  4. (Re-birth) dead cell with 3 live neighbors comes alive

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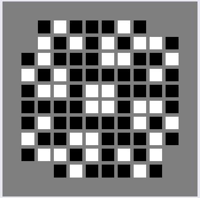

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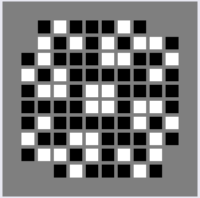

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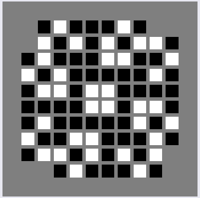

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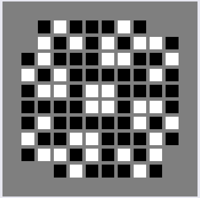

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Encoded as Skolem function existence and synthesis problem

- Let $\mathbf{X}$ be current position, $\mathbf{Y}$ be previous position and $T(\mathbf{X}, \mathbf{Y})$ be transition function
- Then GoE does not exist iff $\forall \mathbf{X} \exists \mathbf{Y}\ T(\mathbf{X}, \mathbf{Y})$ is satisfiable!

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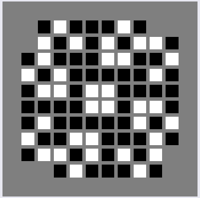

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- Then GoE does not exist iff $\forall \mathbf{X} \exists \mathbf{Y}\ T(\mathbf{X}, \mathbf{Y})$ is satisfiable!
- A witness that GoE does not exist is a Skolem function for $\mathbf{Y}$.

Application Domain 2: Games and planning as QBF

- $\forall \mathbf{X} \exists \mathbf{Y}\ T(\mathbf{X}, \mathbf{Y})$ has two alternating blocks of quantifiers: 2-QBF. In general, can have many!

Quantified Boolean Formula (QBF) or QSAT: Essentially SAT + chunks of quantifiers

$$\forall \mathbf{X}_1 \exists \mathbf{Y}_1 \forall \mathbf{X}_2 \exists \mathbf{Y}_2 \ldots \forall \mathbf{X}_k \exists \mathbf{Y}_k \varphi$$

where φ is a Quantifier-free Boolean Formula,
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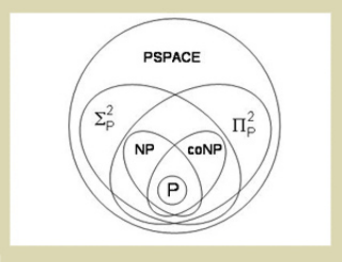
- A rich theoretical history.
  - Textbook PSPACE-complete problem.



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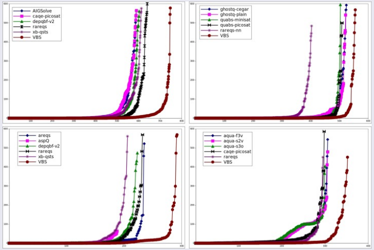
- Huge advances in tools! `https://www.qbflib.org`



Luca Pulina, Martina Seidl: The 2016 and 2017 QBF solvers evaluations (QBFEVAL'16 and QBFEVAL'17). Artif. Intell. 274: 224-248 (2019)

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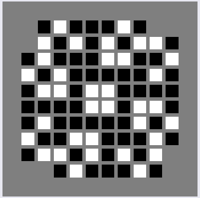

Smallest GoE: found by QBFsolver(2011)

Christiaan Hartman, Marijn Heule, Kees Kwekkeboom, Alain Noels: Symmetry in Gardens of Eden. Electron. J. Comb. 20(3): P16 (2013)

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- Any 2-player game can be coded as QBF
  - Skolem functions are winning strategies of Player 2 ($\exists$-player)!

Ankit Shukla, Armin Biere, Luca Pulina, Martina Seidl: A Survey on Applications of Quantified Boolean Formulas. ICTAI 2019: 78-84

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Is it the case that for every first move of P1 there exists a first move of P2, s.t for every second move of P1 there exists a second move of P2... s.t. P2 can win!?

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  - Skolem functions are winning strategies of Player 2 ($\exists$-player)!
- Many applications of QBF that we dont have time to go into!

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Application Domain 2: Games and planning with QBF and Skolem fns

Conformant or Conditional Planning in AI

Rintanen, J. 1999. Constructing conditional plans by a theorem-prover. Journal of Artificial Intelligence Research 10:323-352.

- Given set S of states, I, G, formulas over S defining initial and goal states and a set of non-det actions A,
  - does there exist a plan (seq of actions), s.t., for all possible contingencies (initial states and nondet choices), there exist an execution (seq of states) that reaches the goal state.

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More Planning to QBF approaches:

- Used to reduce size of encoding rather than uncertainty; Arbitrary Quantifier Alternation.

Michael Cashmore, Maria Fox, Enrico Giunchiglia: Partially Grounded Planning as Quantified Boolean Formula. ICAPS 2013

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Bottomline

Synthesizing Skolem functions synthesizes the plans in all these cases!

Application Domain 3: Reactive Synthesis

Boolean functional synthesis can help reactive synthesis too!

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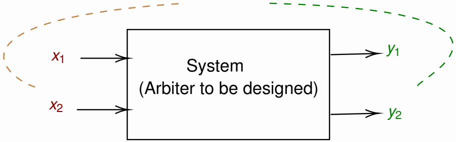
Linear Temporal Logic (LTL) spec

$\mathbf{G}(x_1 \Rightarrow \mathbf{F}y_1) \wedge \mathbf{G}(x_2 \Rightarrow \mathbf{F}y_2) \wedge \mathbf{G}(\neg y_1 \vee \neg y_2)$



- Specification has temporal operators

Application Domain 3: Reactive Synthesis

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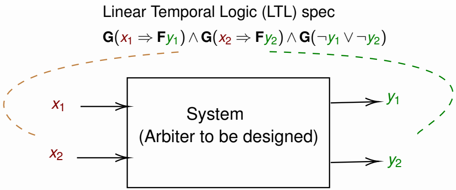
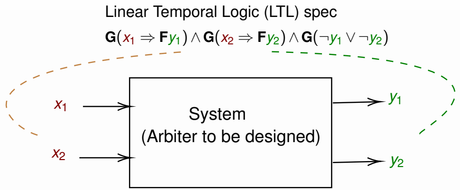


- Specification has temporal operators
- **G**: at **all** times; **F**: **now** or **in future**

Application Domain 3: Reactive Synthesis

Boolean functional synthesis can help reactive synthesis too!



- Specification has temporal operators
- **G**: at **all** times; **F**: **now** or **in future**
- At **all** times
  - If a request comes on x_i, a grant goes on y_i **then or later**.
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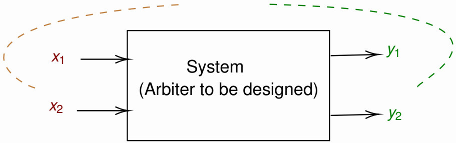

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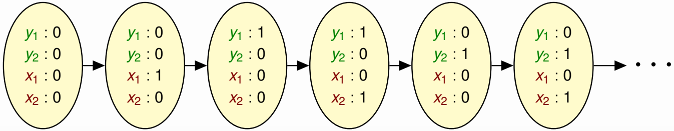

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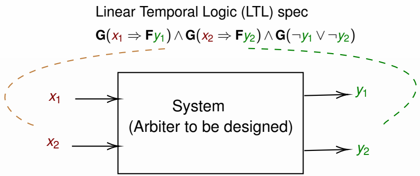

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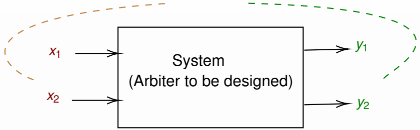

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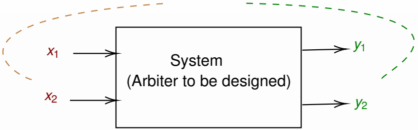

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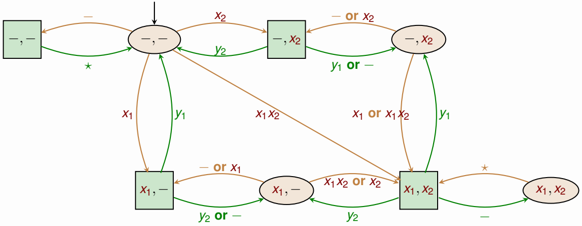
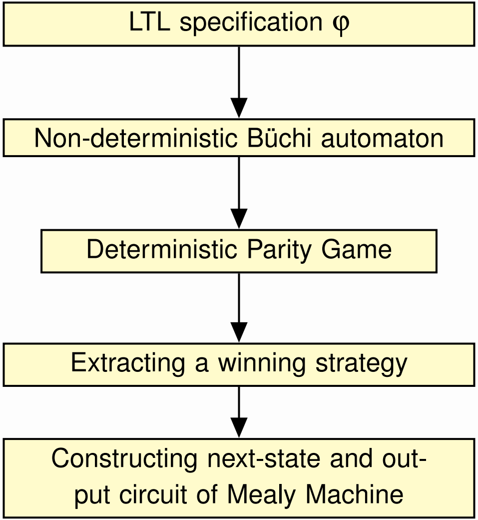
Game graph:



Application Domain 3: Reactive Synthesis

Basic Steps in Synthesis from LTL



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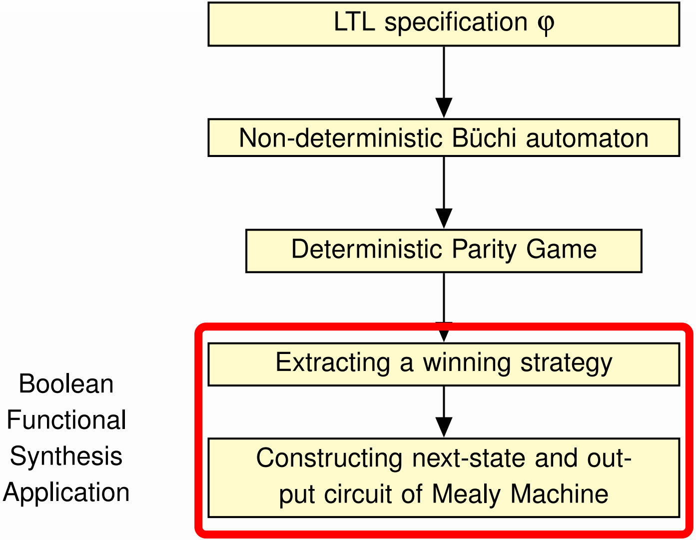

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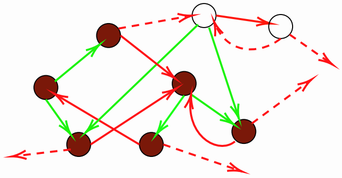

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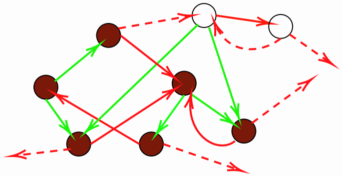

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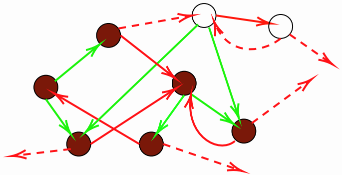

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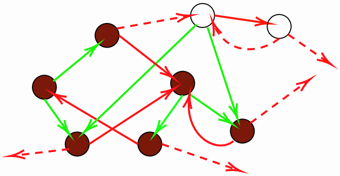

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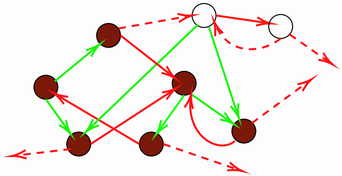

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  - Not always satisfiable

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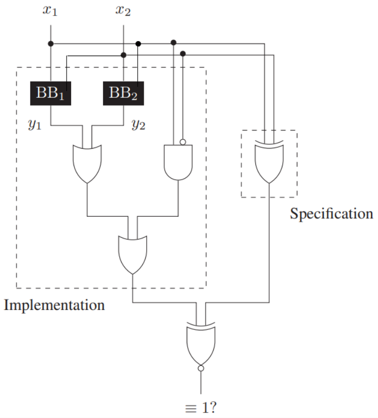

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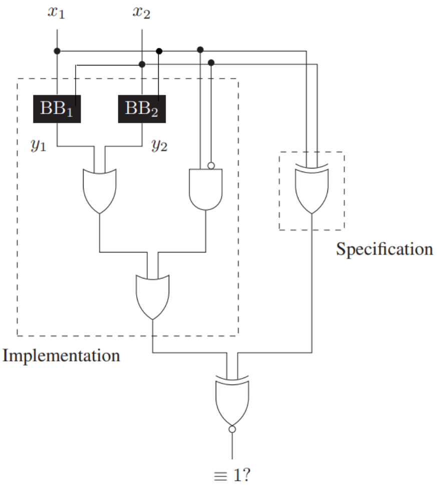

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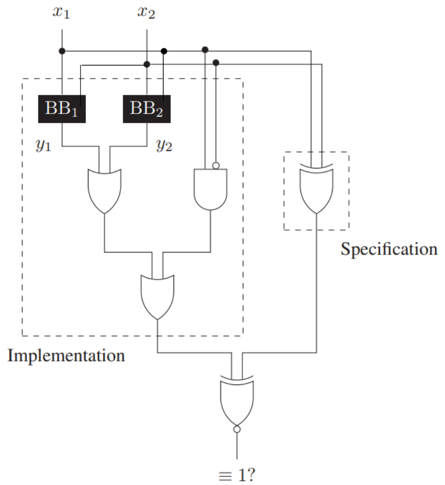

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$$\forall x_1, x_2 \exists y_1 y_2 \neg(((y_1 \vee y_2) \vee (x_1 \wedge \neg x_2)) \oplus (x_1 \oplus x_2))$$

Image is taken(modified) from Equivalence Checking of Partial Designs Using Dependency Quantified Boolean Formulae, Gitina et al '13
Engineering change order for combinational and sequential design rectification, Jiang et. al'20
Synthesis and optimization of multiple portions of circuits for ECO based on set-covering and QBF formulations, Fujita et al'20

Outline

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Efficient algorithms for Boolean functional synthesis unlikely

A Survey of Existing Techniques

0. Closely related to most general Boolean unifiers
   Boole'1847, Lowenheim'1908, Macii'98
1. Extract Skolem functions from proof of validity of $\forall \mathbf{X} \exists \mathbf{Y} \varphi(\mathbf{X}, \mathbf{Y})$
   Bendetti'05, Jussilla et al.'07, Balabanov et al.'12, Heule et al.'14
   - Efficient if a short proof of validity is found.
   - Does not work if $\forall \mathbf{X} \exists \mathbf{Y} \varphi(\mathbf{X}, \mathbf{Y})$ is not valid !
2. Using templates
   Solar-Lezama et al.'06, Srivastava et al.'13
   - Effective when small set of candidate Skolem functions known.
3. Self-substitution + function composition
   Jiang'09, Trivedi'03
   - Craig Interpolation-based approach.
   - Does not scale well with an increase in Y variables.

Existing Approaches (Cont.)

4. Incremental determinization
   Rabe et al.'17,'18
   – Incrementally adds new constraints to the formula to generate a unique Skolem function.
5. Quantifier instantiation techniques in SMT solvers
   Barrett et al.'15, Bierre et al.'17
   – Works even for bit-vector and other theories.
6. Input/output component separation
   Chakraborty et al.'18
   – View specification as made of input and output components.
   – Alternate analysis of each component to generate decision lists.
7. Synthesis from and as ROBDDs
   – Kukula et al.'00, Kuncak et al.'10, Fried et al.'16, Tabajara et al.'17

Existing Approaches (Cont.)

8. Synthesis from special normal forms: The power of Knowledge Compilation!
   - Synthesis negation normal forms (SynNNF)
     Akshay et al.'19
   - The ultimate normal form Shah et al.'21
9. Counter-example guided Skolem function generation
   - Start with over-approximation of Skolem functions + refine
     John et al.'15, Akshay et al.'17,'18,'20
10. Data-driven Skolem function synthesis
    - Machine-learn Skolem function + MaxSat-based iterative repair
      Golia et al.'20, '21

The last two fall into paradigm of Get Skolem function candidate + check + repair

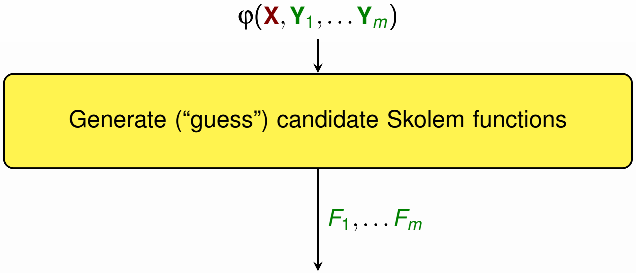
Our focus in the deep-dive: The last three approaches!

Outline

A Guess-Check-Repair Approach

$$\varphi(\mathbf{X}, \mathbf{Y}_1, \ldots \mathbf{Y}_m)$$

A Guess-Check-Repair Approach



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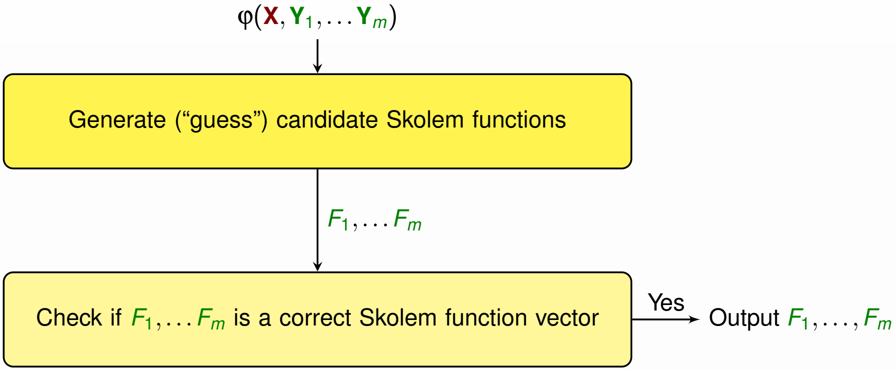

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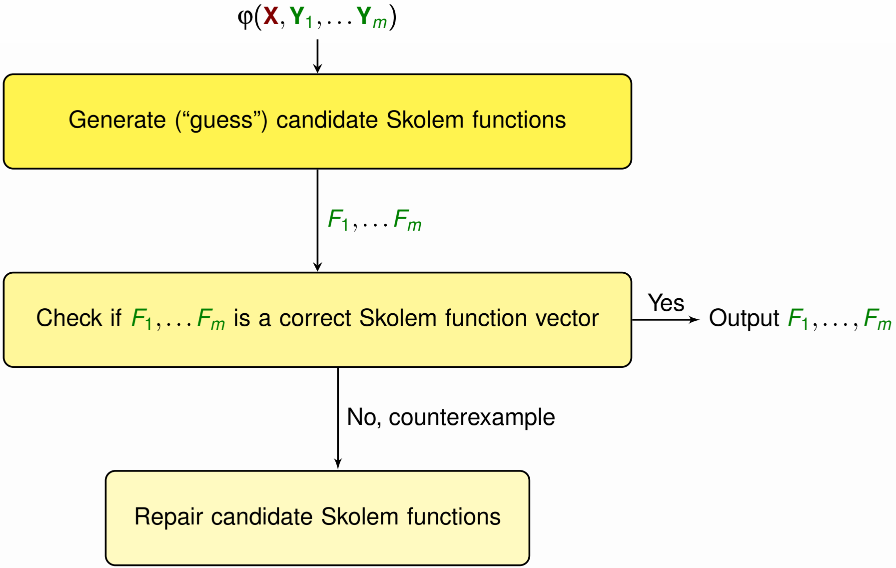

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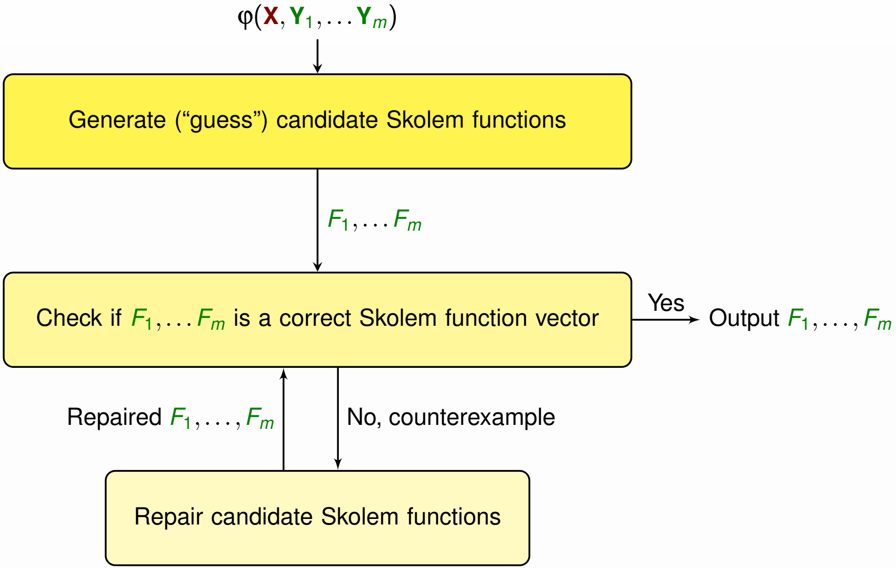

“Guess”-ing candidate Skolem functions ($|\mathbf{Y}| = 1$)

Find $\mathbf{F}(\mathbf{X})$ such that $\exists \mathbf{y}\ \varphi(\mathbf{X},\mathbf{y}) \equiv \varphi(\mathbf{X},\mathbf{F}(\mathbf{X}))$

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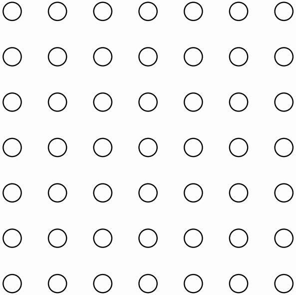

— Set of all valuations of $\mathbf{X}$.

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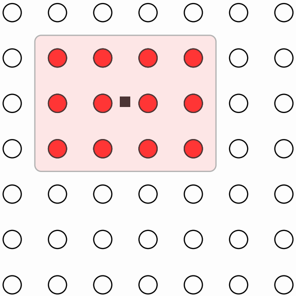

— Can't set $\mathbf{y}$ to 1 to satisfy φ: $\Gamma(\mathbf{X}) \triangleq \neg\varphi(\mathbf{X}, \mathbf{y})[\mathbf{y}1]$

E.g. If $\varphi \equiv (x_1 \vee \mathbf{y}) \wedge (x_1 \vee x_2 \vee \neg\mathbf{y})$, then
$\Gamma(\mathbf{X}) = \neg((x_1 \vee 1) \wedge (x_1 \vee x_2 \vee 0)) = \neg(x_1 \vee x_2) = \neg x_1 \wedge \neg x_2$

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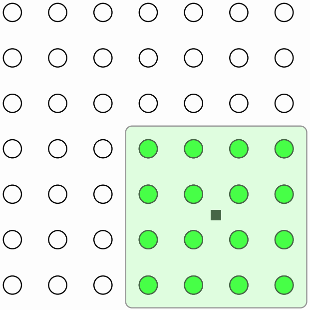

— Can't set $\mathbf{y}$ to 0 to satisfy φ: $\Delta(\mathbf{X}) \triangleq \neg\varphi(\mathbf{X}, \mathbf{y})[\mathbf{y}0]$

E.g. If $\varphi \equiv (x_1 \vee \mathbf{y}) \wedge (x_1 \vee x_2 \vee \neg\mathbf{y})$, then $\Delta(\mathbf{X}) = \neg((x_1 \vee 0) \wedge (x_1 \vee x_2 \vee 1)) = \neg x_1$

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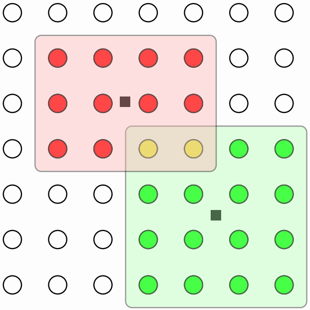

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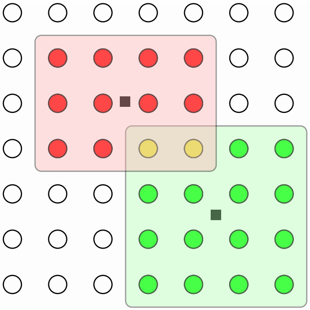

Lemma [Trivedi'03, Jiang'09,Fried et al'16]

Every Skolem function for $\mathbf{y}$ in φ must

- Evaluate to 1 in $(\Delta \setminus \Gamma)$ and to 0 in $(\Gamma \setminus \Delta)$
- Be an **interpolant** of $(\Delta \setminus \Gamma)$ and $(\Gamma \setminus \Delta)$

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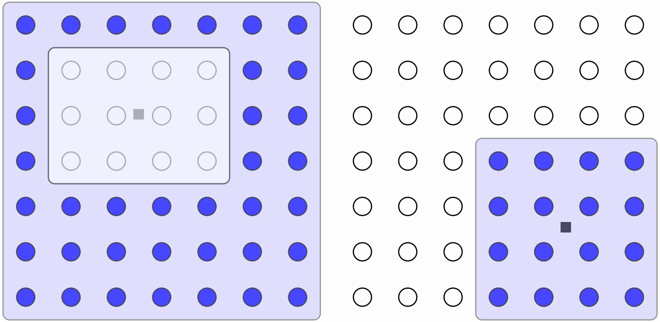

— Specific interpolants of $(\Delta \setminus \Gamma)$ & $(\Gamma \setminus \Delta)$

- $\neg\Gamma \triangleq \varphi(\mathbf{X}, \mathbf{y})[\mathbf{y}1] \equiv \varphi(\mathbf{X}, 1)$
- $\Delta \triangleq \neg\varphi(\mathbf{X}, y)[\mathbf{y}0] \equiv \neg\varphi(\mathbf{X}, 0)$.

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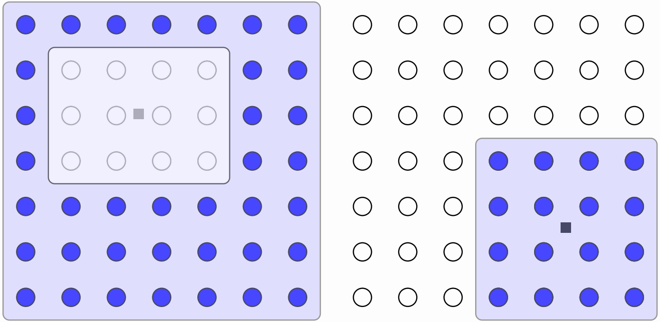

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- $\neg\Gamma \triangleq \varphi(\mathbf{X}, \mathbf{y})[\mathbf{y}1] \equiv \varphi(\mathbf{X}, 1)$: Easy solution for 1 output var
- $\Delta \triangleq \neg\varphi(\mathbf{X}, y)[\mathbf{y}0] \equiv \neg\varphi(\mathbf{X}, 0)$.

“Guess”-ing Game: ($|\mathbf{Y}| \geq 2$)

Suppose relational spec is $\varphi(\mathbf{X}, y_1, \boxed{\mathbf{Y}_{2..m}})$

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  - Repeat above steps ...

"Guess"-ing Game: ($|\mathbf{Y}| \geq 2$)

Suppose relational spec is $\varphi(\mathbf{X}, y_1, \boxed{\mathbf{Y}_{2..m}})$

- Skolem function for $\boxed{\mathbf{Y}_{2..m}}$ depends on that for y_1 in general
- For what values of $\mathbf{X}$ can we not set y_1 to 1 (or 0)?
  - $\Gamma^{y_1}(\mathbf{X}) = \neg\exists\, \mathbf{Y}_{2..m}\ \varphi(\mathbf{X}, 1, \boxed{\mathbf{Y}_{2..m}})$
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- From $\Gamma^{y_1}(\mathbf{X})$ and $\Delta^{y_1}(\mathbf{X})$, find Skolem function $F_1(\mathbf{X})$ for y_1
- To find Skolem function for y_2, consider
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Are we done?

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  - Use easily computed approx of $\exists\, \mathbf{Y}_{2..m}\ \varphi(\mathbf{X}, y_1, \boxed{\mathbf{Y}_{2..m}})$?
  - “Guess” $G_1(\mathbf{X})$ as approx of Skolem function $F_1(\mathbf{X})$?
  - Repair “guess” if needed
- To find Skolem function for y_2, consider
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Linearly order outputs: $y_1 \prec y_2 \prec \cdots \prec y_m$

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- Generate Skolem functions for 1-output spec
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- #gates $\leq \sum_{i=1}^{m}$ #gates(G_i) $+2m$
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Sufficient to compute the G_i functions

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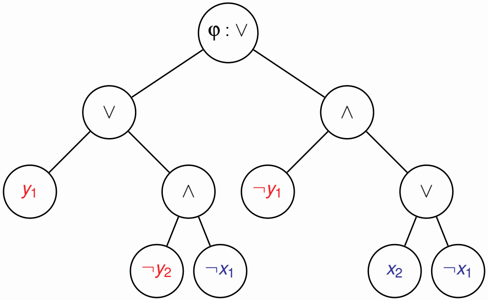

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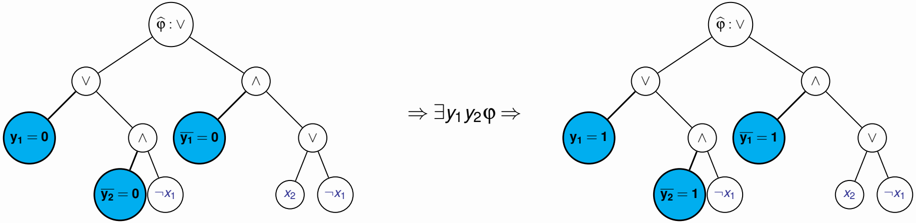
Replace $\neg y_i$ at leaves with fresh variables $\overline{y_i}$ and call the "new" formula $\widehat{\varphi}$.



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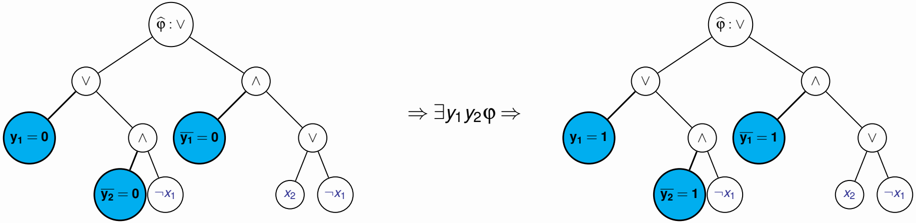

- $\widehat{\varphi}(x_1 \ldots x_n, \overbrace{0..0}^{i}, y_{i+1} \ldots y_m, \overbrace{0..0}^{i}, \neg y_{i+1} \ldots \neg y_m) \Rightarrow \exists y_1 \ldots y_i\ \varphi(\ldots)$
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Checking correctness of "guess"-ed Skolem functions

Given candidate Skolem functions $F_1, \ldots F_m$,

$$\text{Is } \forall \mathbf{X} \left(\exists \mathbf{Y} \varphi(\mathbf{X}, \mathbf{Y}) \Leftrightarrow \varphi(\mathbf{X}, \mathbf{F}(\mathbf{X}) \right) \text{ ?}$$

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  - σ is **counterexample** to the claim that $F_1, \ldots F_m$ is a correct Skolem function vector

Counterexample Generalization

Recall: Skolem functions guessed from approximations of

$\exists y_{i+1} \ldots \exists y_m \, \varphi(\mathbf{X}, x_1, \ldots x_{i-1}, y_i, y_{i+1}, \ldots y_m)$

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$\exists y_{i+1} \ldots \exists y_m \varphi(\mathbf{X}, x_1, \ldots x_{i-1}, y_i, y_{i+1}, \ldots y_m)$

- Let $\exists y_{i+1} \ldots \exists y_m \varphi(\mathbf{X}, \mathbf{Y}) \Rightarrow \Theta_i(\mathbf{X}, x_1, \ldots x_{i-1}, y_i)$
- Let $\delta_i = \neg\Theta_i|_{y_i=0}$; $\quad \gamma_i = \neg\Theta_i|_{y_i=1}$
- Initial guess $G_i(\mathbf{X}, x_1, \ldots x_{i-1}) \in \{\delta_i, \neg\gamma_i\}$ **... 1-sided error**
  - $G_i = \delta_i$ **cannot err** if it evaluates to 1
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Generalized counterexample

Given $\sigma \models \varepsilon(\mathbf{X}, \mathbf{Y}, \mathbf{Y}')$ and δ_i, γ_i for $1 \leq i \leq m$

Find function $\mu(\mathbf{X}, x_1, \ldots x_{j-1})$ for some $j \in \{1, \ldots m\}$ s.t.

- $\sigma \models \mu$ **... μ generalizes σ**
- $\mu \Rightarrow \gamma_j \wedge \delta_j$
  - $\Rightarrow \forall y_j \ldots \forall y_m \neg\varphi(\mathbf{X}, x_1, \ldots x_{j-1}, y_j, y_{j+1}, \ldots y_m)$
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Must ensure that $(\mathbf{X}, G_1, \ldots G_{j-1})$ never evaluates to π

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Expansion-based repair

Simple argument for termination – expansions can't go on forever

Deep Dive 2: Knowledge compilation for Boolean Functional Synthesis

Knowledge Compilation for Synthesis

Our Definition

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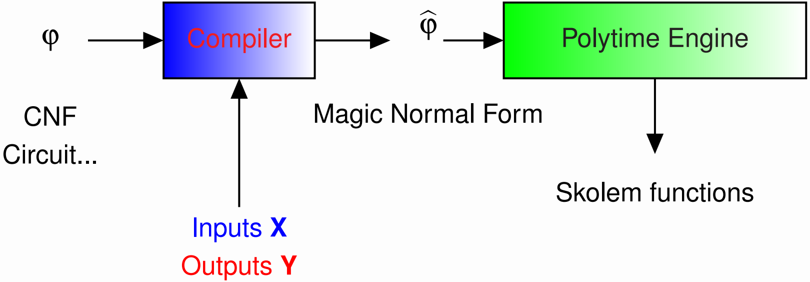

Recall from the previous Deep-Dive

For solving the Skolem function synthesis problem, it suffices to

1. Generate Skolem functions for only 1-output specs
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Does there exist a form of the specification where this HARD question is EASY?

Towards a normal form for efficient synthesis

- Represent $\varphi(x_1, .., x_n, y_1, .., y_m)$ NNF DAG
  - Boolean circuit, $\wedge$ and $\vee$ at internal nodes, $\neg$ only at leaves

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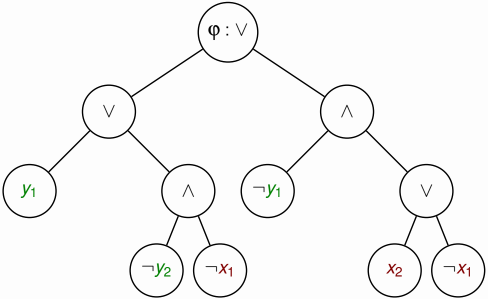

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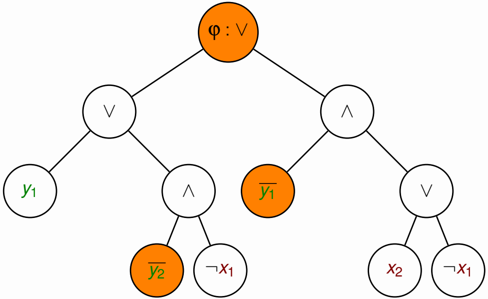

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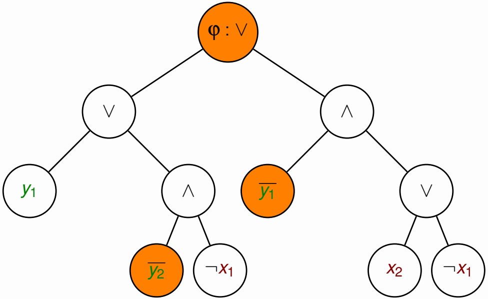
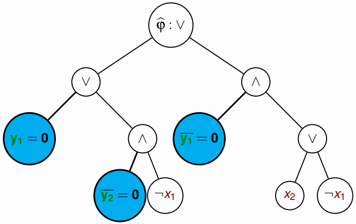


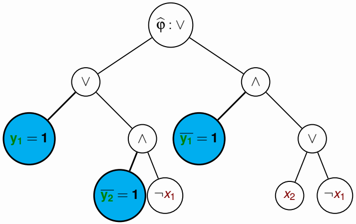
Positive form of specification: $\widehat{\varphi}(\{x_1, \ldots x_n\}, \{y_1, \ldots, y_m, \overline{y_1}, \ldots \overline{y_m}\})$

- Monotone w.r.t all y_i and $\overline{y_i}$

Simple properties of the positive form $\widehat{\varphi}$



$\Rightarrow \exists y_1 y_2 \varphi \Rightarrow$



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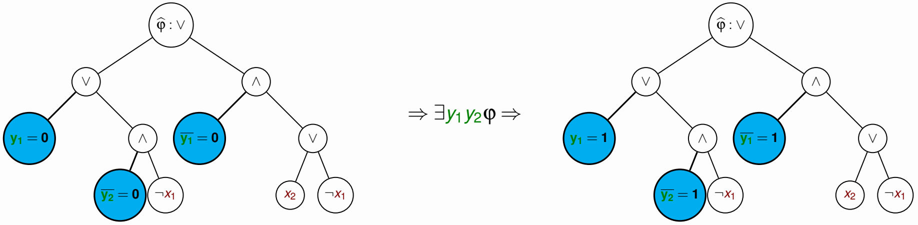

- $\widehat{\varphi}(x_1 \ldots x_n, \overbrace{0..0}^{i}, y_{i+1} \ldots y_m, \overbrace{0..0}^{i}, \neg y_{i+1} \ldots \neg y_m) \Rightarrow \exists y_1 \ldots y_i \; \varphi(\ldots)$
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Let us take the first output: $\exists y_1 \varphi(\mathbf{X}, \mathbf{Y}) \;\Rightarrow\; \widehat{\varphi}\,|_{y_1=1,\overline{y_1}=1}$

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We can now generalize this to more outputs

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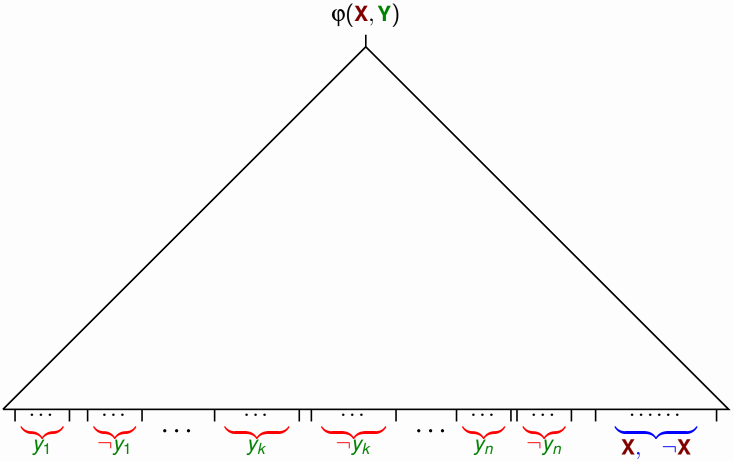
- We want to ensure the positive form does not "behave" as $y_i \wedge \overline{y_i}$ for any i.
- What representation of the specification φ ensures this?

A simple yet special Normal Form

Decomposable Negation Normal Form (DNNF): Forbidden **structure**



Adnan Darwiche: Decomposable negation normal form. J. ACM 48(4): 608-647 (2001)

A simple yet special Normal Form

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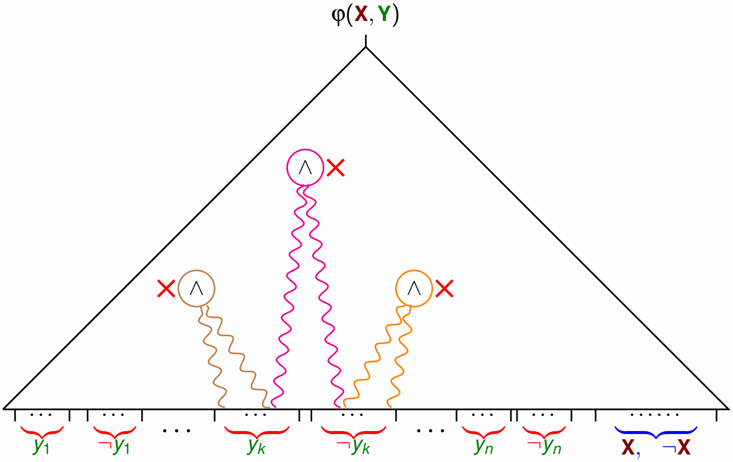

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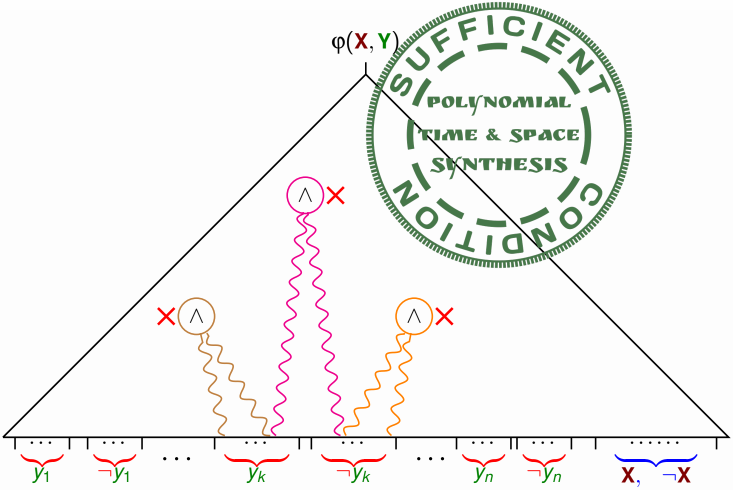
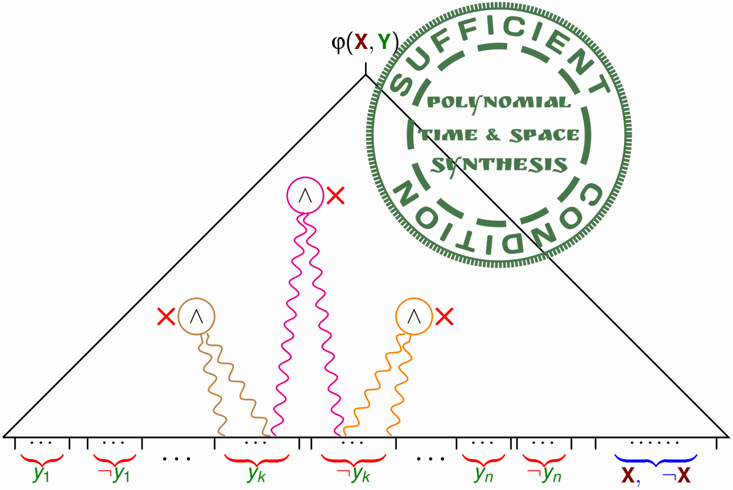


Adnan Darwiche: Decomposable negation normal form. J. ACM 48(4): 608-647 (2001)

A simple yet special Normal Form

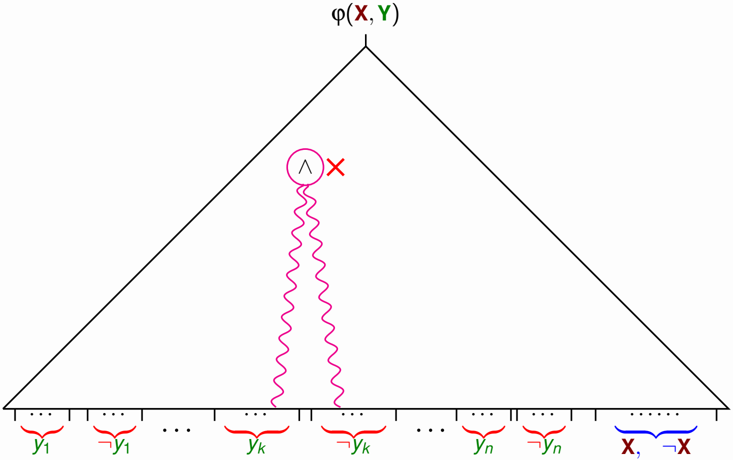
Decomposable Negation Normal Form (DNNF): Forbidden **structure**

DNNF has many other nice properties. Well-studied in the KR community!



Surely, we can do better!

Weak DNNF (wDNNF): Forbidden **structure**



S. Akshay, Supratik Chakraborty, Shubham Goel, Sumith Kulal, Shetal Shah: What's Hard About Boolean Functional Synthesis? CAV (1) 2018: 251-269

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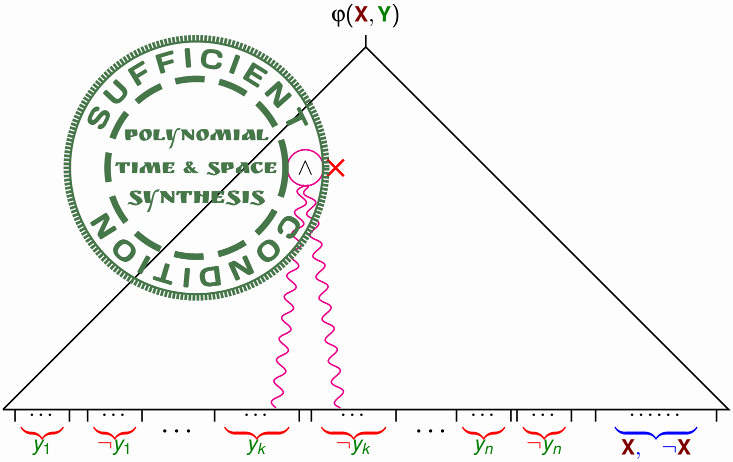
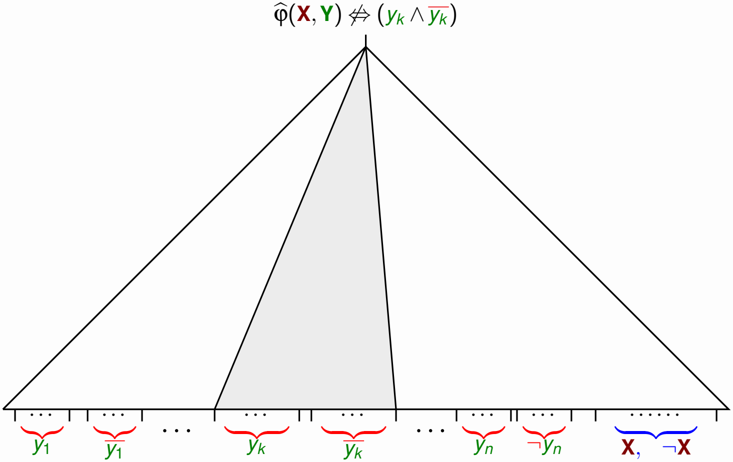


Exploit the property of the reduct!

Synthesis Negation Normal Form (SynNNF): Forbidden **semantics**



S. Akshay, J. Arora, S. Chakraborty, S. Krishna, D. Raghunathan, S. Shah: Knowledge Compilation for Boolean Functional Synthesis. FMCAD 2019: 161-169

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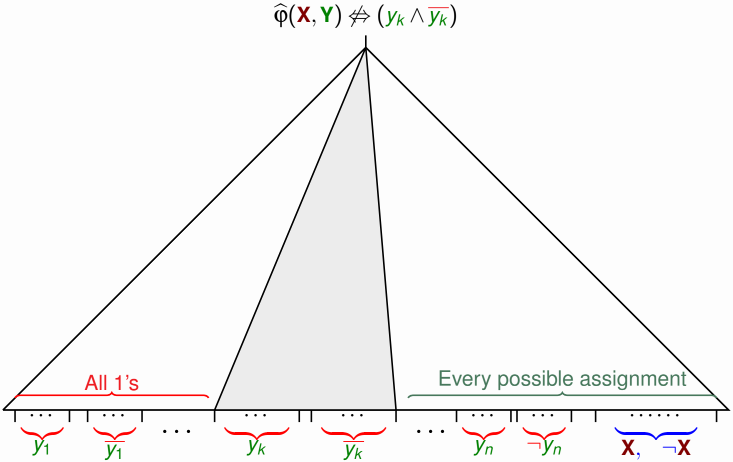
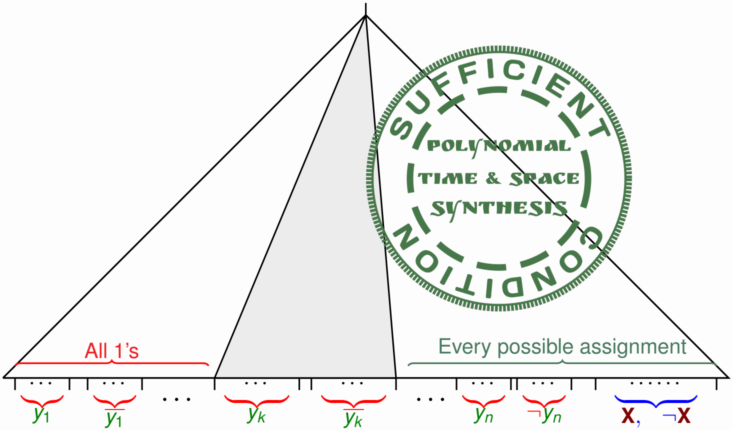


Synthesis Negation Normal Form (SynNNF): Forbidden **semantics**

$\widehat{\varphi}(\mathbf{X}, \mathbf{Y}) \not\Rightarrow (y_k \wedge \overline{y_k})$



Skolem fn for y_i (in terms of $y_{i+1}, \ldots y_m, \mathbf{X}$)

- $\exists y_1, \ldots y_{i-1}\ \varphi(\mathbf{X}, y_1, \ldots y_{i-1}, 1, y_{i+1}, \ldots y_m)$

SynNNF: A negation normal form for efficient synthesis

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Poly-time/sized Skolem functions!

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Observations:

- Not purely structural restriction on representation of φ

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Poly-time/sized Skolem functions!

Observations:

- Not purely structural restriction on representation of φ
- Reminiscent of Deterministic DNNF (dDNNF)
  - For every $\vee$ node representing $\varphi_1 \vee \varphi_2$, require $\varphi_1 \wedge \varphi_2 = \bot$.

Comparing the Normal Forms

- Every wDNNF, DNNF circuit is also in SynNNF.
- Every FBDD, ROBDD can be compiled in linear time to SynNNF.

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Punchline!

SynNNF is exponentially more succinct than DNNF/dDNNF

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SynNNF is exponentially more succinct than DNNF/dDNNF, which are themselves exponentially more succinct than ROBDDs/FBDD.

What more can we do?

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Can we get necessary & sufficient condition?

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Characterizing poly-time and poly-size BFnS

Does there exist a ”semantically universal” class $\mathcal{C}^\star$ of ckts s.t.:

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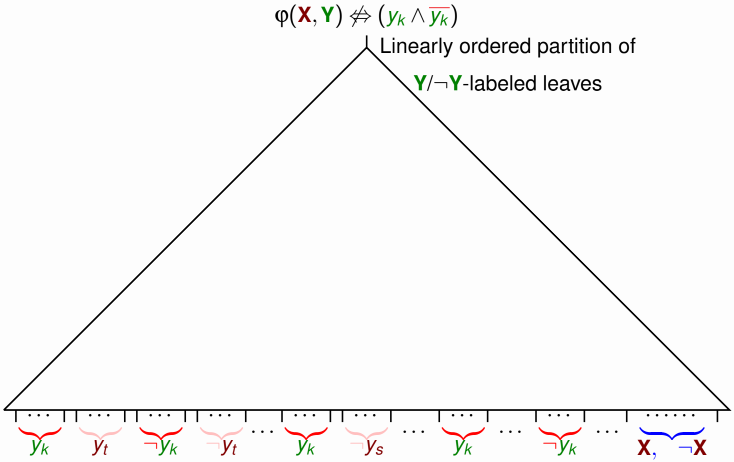
Surprise!

Yes, there exists such a class! Subset-And-Unrealizable Normal Form (SAUNF)

P. Shah, A. Bansal, S. Akshay, S. Chakraborty: A Normal Form Characterization for Efficient Boolean Skolem Function Synthesis. LICS 2021: 1-13

SAUNF: A Very Special Normal Form

Generalizing forbidden **semantics** of SynNNF



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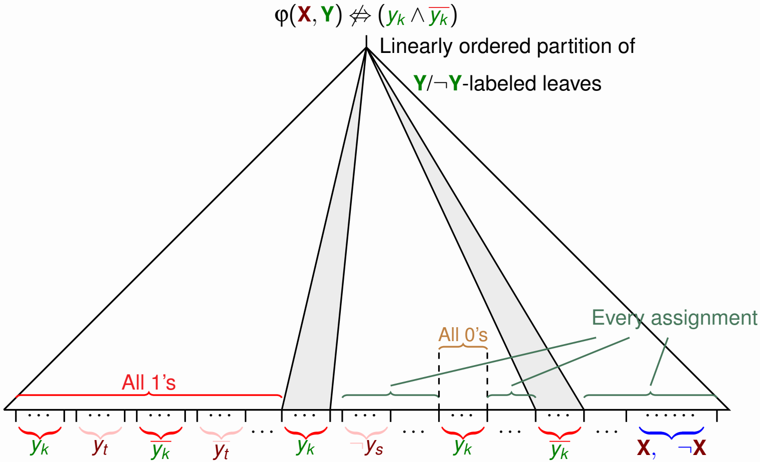

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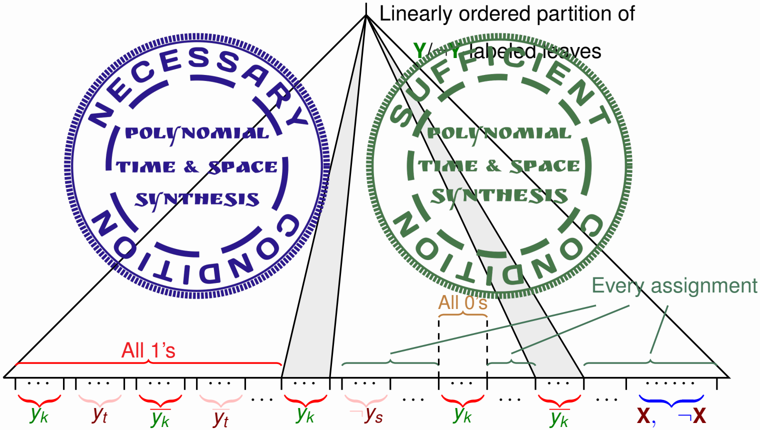

More properties and structure

Checking if a given specification is in SynNNF/SAUNF

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- Algorithms and Prototype implementations exist. (e.g., C2Syn)
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  - Unavoidable due to hardness results

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Algorithms for compositions and operations

Given $\varphi_1(\mathbf{X}, \mathbf{Y})$ and $\varphi_2(\mathbf{X}, \mathbf{Y})$ in SynNNF/SAUNF

- Computing $\varphi_1 \vee \varphi_2$ in SynNNF/SAUNF takes constant time.

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- Existential quantification is easy.

End of this deep dive

Some takeaways

- Nice normal forms exist for Boolean Functional Synthesis!

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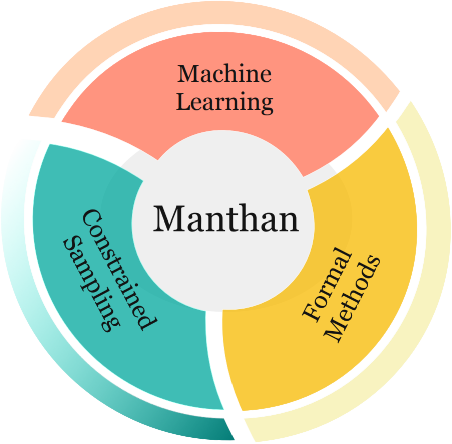
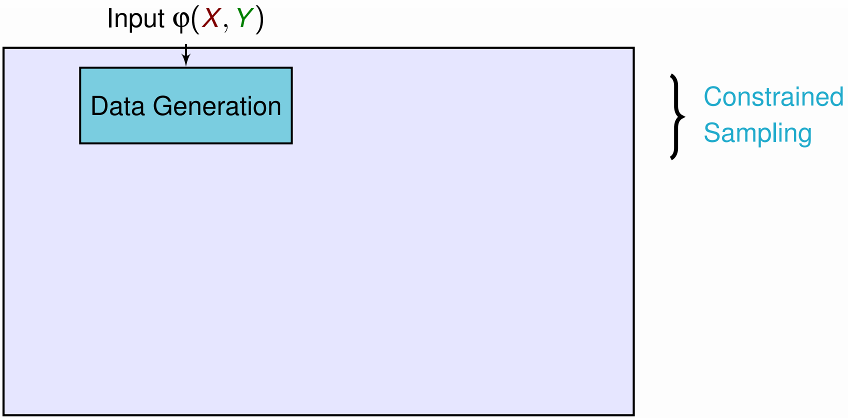
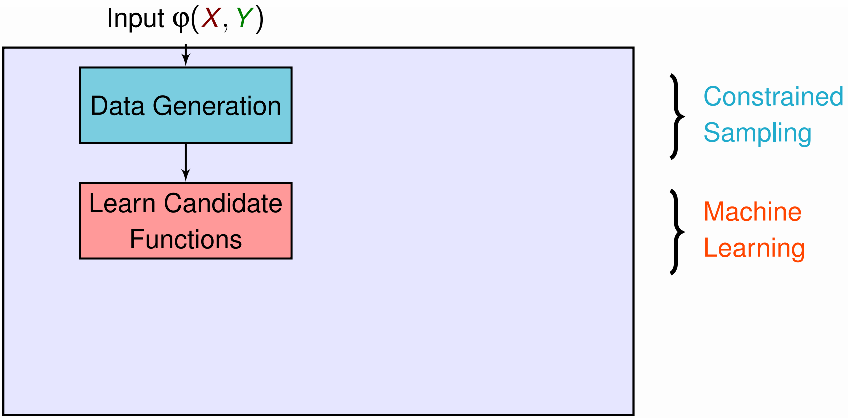
- Explains performance of existing tools on some benchmarks.
- More in concluding remarks...

Another Flavour of Guess-Check-Repair

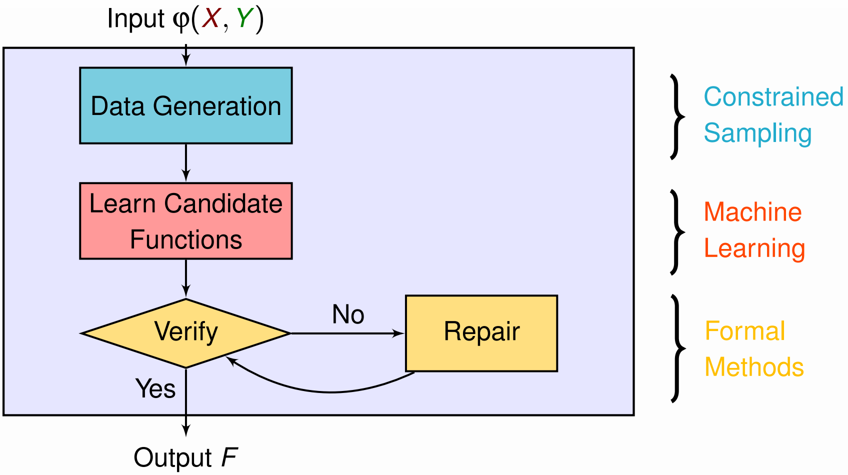


A data-driven approach for Skolem function synthesis


Input φ(X, Y)
Data Generation
Constrained
Sampling


Input φ(X, Y)
Data Generation
Learn Candidate Functions
Constrained Sampling
Machine Learning

Manthan

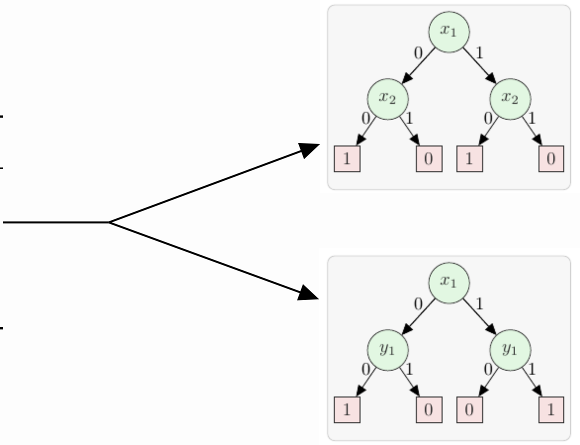

Input φ(X, Y)
Data Generation
Learn Candidate Functions
Verify
No
Repair
Yes
Output F
Constrained Sampling
Machine Learning
Formal Methods

$\varphi(x_1, x_2, y_1, y_2)$ ⟶

x_1	x_2	y_1	y_2
0	0	1	0
0	1	0	1
1	0	1	1
1	1	0	0

Learn Candidate Functions

x_1	x_2	y_1	y_2
0	0	1	0
0	1	0	1
1	0	1	1
1	1	0	0



$p_1 := (\neg x_1 \wedge \neg x_2)$,
$p_2 := (x_1 \wedge \neg x_2)$
f_1 = if p_1 then 1
elif p_2 then 1
else 0

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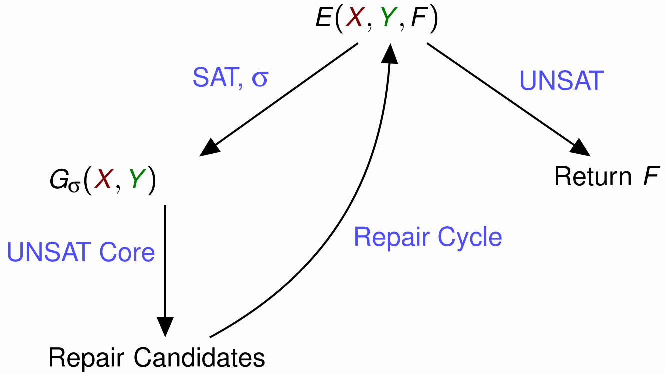
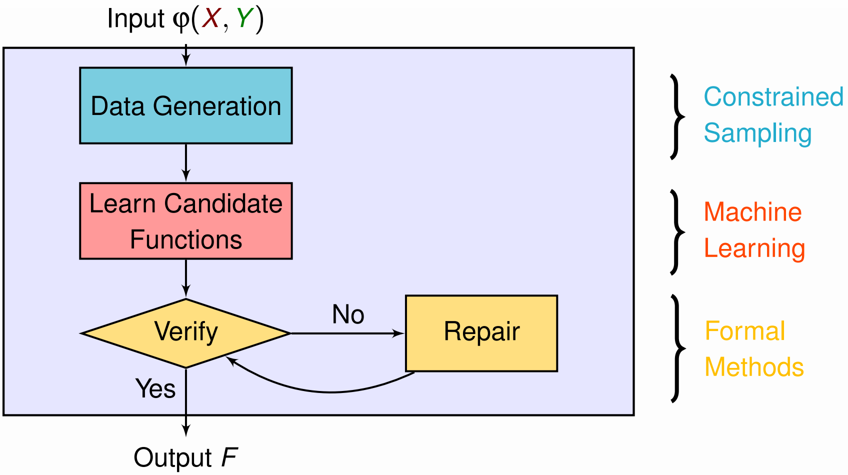
$p_2 := (x_1 \wedge y_1)$
f_1 = if p_1 then 1
elif p_2 then 1
else 0


E(X, Y, F)
SAT, σ
UNSAT
G_σ(X, Y)
Return F
UNSAT Core
Repair Cycle
Repair Candidates


Input φ(X, Y)
Data Generation
Learn Candidate Functions
Verify
No
Repair
Yes
Output F
Constrained Sampling
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Formal Methods

Data Generation

Potential Strategy: Randomly sample satisfying assignment of $\varphi(X, Y)$.

Challenge: Multiple valuations of y_1, y_2 for same valuation of x_1, x_2.

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$$\varphi(x_1, x_2, y_1, y_2) : (x_1 \vee x_2 \vee y_1) \wedge (\neg x_1 \vee \neg x_2 \vee \neg y_2)$$

x_1	x_2	y_1	y_2
0	0	1	0/1
0	1	0/1	0/1
1	0	0/1	0/1
1	1	0/1	0

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$$\varphi(x_1, x_2, y_1, y_2) : (x_1 \vee x_2 \vee y_1) \wedge (\neg x_1 \vee \neg x_2 \vee \neg y_2)$$

x_1	x_2	y_1	y_2
0	0	1	0/1
0	1	0/1	0/1
1	0	0/1	0/1
1	1	0/1	0

Uniform Sampler →

x_1	x_2	y_1	y_2
0	0	1	1
0	1	0	1
1	0	0	1
1	1	0	0

Data Generation

$$\varphi(x_1, x_2, y_1, y_2) : (x_1 \lor x_2 \lor y_1) \land (\neg x_1 \lor \neg x_2 \lor \neg y_2)$$

x_1	x_2	y_1	y_2
0	0	1	0/1
0	1	0/1	0/1
1	0	0/1	0/1
1	1	0/1	0

Uniform Sampler →

x_1	x_2	y_1	y_2
0	0	1	1
0	1	0	1
1	0	0	1
1	1	0	0

- Possible Skolem functions:
  - $f_1(x_1, x_2) = \neg(x_1 \lor x_2)$
  - $f_2(x_1, x_2) = \neg(x_1 \land x_2)$

Data Generation

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x_1	x_2	y_1	y_2
0	0	1	0/1
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x_1	x_2	y_1	y_2
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  - $f_2(x_1, x_2) = \neg(x_1 \wedge x_2)$ $\quad f_2(x_1, x_2) = \neg x_1$ $\quad f_2(x_1, x_2) = \neg x_2$ $\quad f_2(x_1, x_2) = 0$

Data Generation

$$\varphi(x_1, x_2, y_1, y_2) : (x_1 \vee x_2 \vee y_1) \wedge (\neg x_1 \vee \neg x_2 \vee \neg y_2)$$

x_1	x_2	y_1	y_2
0	0	1	0/1
0	1	0/1	0/1
1	0	0/1	0/1
1	1	0/1	0

Magical Sampler →

x_1	x_2	y_1	y_2
0	0	1	0
0	1	1	0
1	0	1	0
1	1	1	0

- Possible Skolem functions:
  - $f_1(x_1, x_2) = \neg(x_1 \vee x_2)$ $\quad f_1(x_1, x_2) = \neg x_1$ $\quad f_1(x_1, x_2) = \neg x_2$ $\quad f_1(x_1, x_2) = 1$
  - $f_2(x_1, x_2) = \neg(x_1 \wedge x_2)$ $\quad f_2(x_1, x_2) = \neg x_1$ $\quad f_2(x_1, x_2) = \neg x_2$ $\quad f_2(x_1, x_2) = 0$

Weighted Sampling to Rescue

- $W : X \cup Y \mapsto [0,1]$

- The probability of generation of an assignment is proportional to its weight.

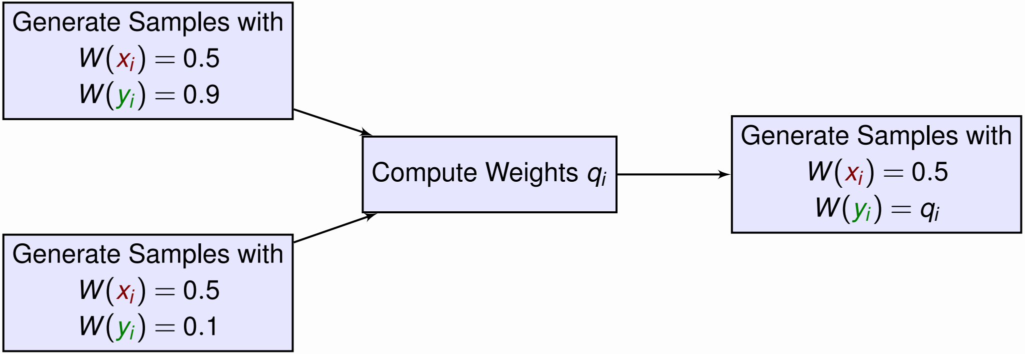
$$W(\sigma) = \prod_{\sigma(z_i)=1} W(z_i) \prod_{\sigma(z_i)=0} (1 - W(z_i))$$

- Example: $W(x_1) = 0.5 \quad W(x_2) = 0.5 \quad W(y_1) = 0.9 \quad W(y_2) = 0.1$
  $\sigma_1 = \{x_1 \mapsto 1, x_2 \mapsto 0, y_1 \mapsto 0, y_2 \mapsto 1\}$

$$W(\sigma_1) = 0.5 \times (1 - 0.5) \times (1 - 0.9) \times 0.1 = 0.0025$$

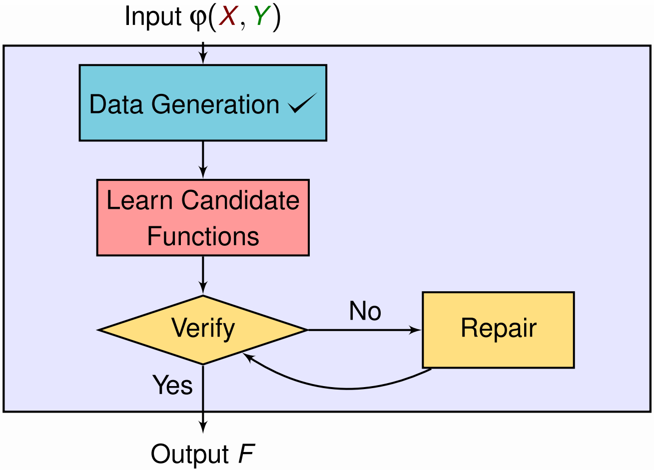
- Uniform sampling is a special case where all variables are assigned weight of 0.5.

Data Generation



Different Sampling Strategies

- Knowledge representation based techniques
  (Yuan,Shultz, Pixley,Miller,Aziz 1999)
  (Yuan,Aziz, Pixley,Albin, 2004)
  (Kukula and Shiple, 2000)
  (Sharma, Gupta, Meel, Roy, 2018)
  (Gupta, Sharma, Meel, Roy, 2019)
- Hashing based techniques
  (Chakraborty, Meel, and Vardi 2013, 2014,2015)
  (Soos, Meel, and Gocht 2020)
- Mutation based techniques
  (Dutra, Laeufer, Bachrach, Sen, 2018)
- Markov Chain Monte Carlo based techniques
  (Wei and Selman,2005)
  (Kitchen,2010)
- Constraint solver based techniques
  (Ermon, Gomes, Sabharwal, Selman,2012)
- Belief networks based techniques
  (Dechter, Kask, Bin, Emek,2002)
  (Gogate and Dechter,2006)


Input φ(X, Y)
Data Generation ✓
Learn Candidate Functions
Verify
No
Repair
Yes
Output F

Learn Candidate Function: Decision Tree Classifier

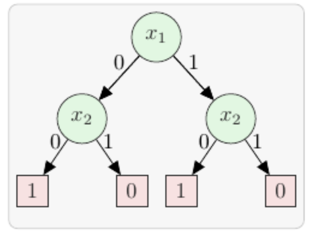
$$\varphi(x_1, x_2, y_1, y_2) : (x_1 \vee x_2 \vee y_1) \wedge (\neg x_1 \vee \neg x_2 \vee \neg y_2)$$

- To learn y_2
  - Feature set: valuation of x_1, x_2, y_1
  - Label: valuation of y_2
  - Learn decision tree to represent y_2 in terms of x_1, x_2, y_1
- To learn y_1
  - Feature set: valuation of x_1, x_2
  - Label: valuation of y_1
  - Learn decision tree to represent y_1 in terms of x_1, x_2

x_1	x_2	y_1	y_2
0	0	1	0
0	1	0	1
1	0	1	1
1	1	0	0

Learning Candidate Functions

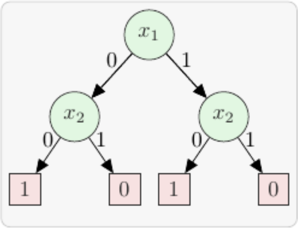
x_1	x_2	y_1	y_2
0	0	1	0
0	1	0	1
1	0	1	1
1	1	0	0



$p_1 := (\neg x_1 \wedge \neg x_2)$,
$p_2 := (x_1 \wedge \neg x_2)$
f_1 = if p_1 then 1
elif p_2 then 1
else 0

Learning Candidate Functions

x_1	x_2	y_1	y_2
0	0	1	0
0	1	0	1
1	0	1	1
1	1	0	0



$p_1 := (\neg x_1 \wedge \neg x_2)$,
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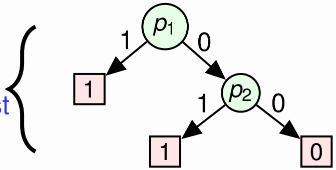
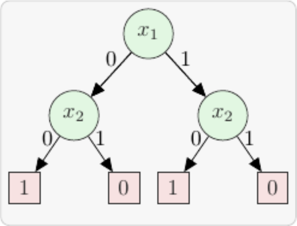
f_1 = if p_1 then 1
elif p_2 then 1
else 0

Can reorder p_1, p_2
Learning one level decision list



What Kind of Learning

x_1	x_2	y_1	y_2
0	0	1	0
0	1	0	1
1	0	1	1
1	1	0	0



$p_1 := (\neg x_1 \wedge \neg x_2)$,
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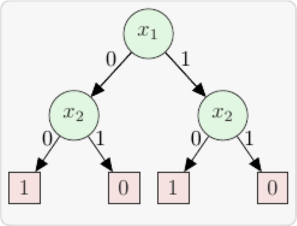
f_1 = if p_1 then 1
elif p_2 then 1
else 0

Learning without Error
Every row is a solution of $\varphi(X, Y)$

Learning with Errors
The data is only a subset of solutions.

x_1	x_2	y_1	y_2
0	0	1	0
0	1	0	1
1	0	1	1
1	1	0	0



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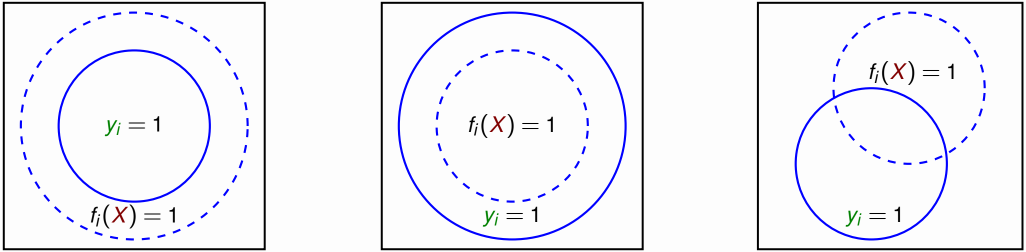
Learning without Error
Every row is a solution of $\varphi(X, Y)$

Learning with Errors
The data is only a subset of solutions.

Learn with Errors: Approximations not Abstractions

Abstraction vs Approximation



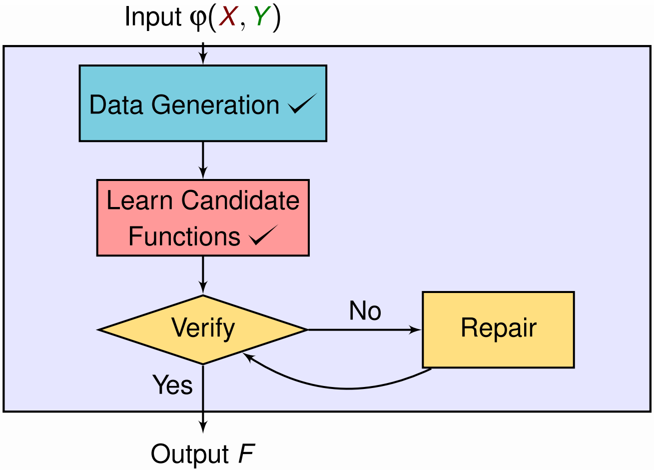
$y_i \rightarrow f_i(X)$ $\qquad$ $f_i(X) \rightarrow y_i$

Abstraction

Approximation

$y_i = 1, f_i(X) = 0$

$y_i = 0, f_i(X) = 1$


Input φ(X, Y)
Data Generation ✓
Learn Candidate Functions ✓
Verify
No
Repair
Yes
Output F

Verification of Candidate Functions

$$E(X, Y, Y') := \varphi(X, Y) \wedge \neg\varphi(X, Y') \wedge (Y' \leftrightarrow F(X))$$

(JSCTA'15)

- If $E(X, Y, Y')$ is UNSAT: $\exists Y \varphi(X, Y) \equiv \varphi(X, F(X))$
  - Return F
- If $E(X, Y, Y')$ is SAT: $\exists Y \varphi(X, Y) \not\equiv \varphi(X, F(X))$
  - Let $\sigma \models E(X, Y, Y')$ be a counterexample to fix.

Repair Candidate Identification

$$E(X, Y, Y') := \varphi(X, Y) \wedge \neg\varphi(X, Y') \wedge (Y' \leftrightarrow F(X))$$

$\sigma \models E(X, Y, Y')$ be a counterexample to fix.

- Let $\sigma := \{x_1 \mapsto 1, x_2 \mapsto 1, y_1 \mapsto 1, y_2 \mapsto 1, y_1' \mapsto 0, y_2' \mapsto 0\}$.
- Potential repair candidates: All y_i where $\sigma[y_i] \neq \sigma[y_i']$.

Repair Candidate Identification

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- Let $\sigma := \{x_1 \mapsto 1, x_2 \mapsto 1, y_1 \mapsto 1, y_2 \mapsto 1, y'_1 \mapsto 0, y'_2 \mapsto 0\}$.
- Potential repair candidates: All y_i where $\sigma[y_i] \neq \sigma[y'_i]$.
- $\varphi(X, Y)$ is Boolean Relation.
  - So it can be $\hat{\sigma} = \{x_1 \mapsto 1, x_2 \mapsto 1, y_1 \mapsto 0, y_2 \mapsto 1, y'_1 \mapsto 0, y'_2 \mapsto 0\}$
  - We would not repair f_1.

Repair Candidate Identification

$$E(X,Y,Y') := \varphi(X,Y) \land \neg\varphi(X,Y') \land (Y' \leftrightarrow F(X))$$

$$\sigma \models E(X,Y,Y') \text{ be a counterexample to fix.}$$

- Let $\sigma := \{x_1 \mapsto 1, x_2 \mapsto 1, y_1 \mapsto 1, y_2 \mapsto 1, y'_1 \mapsto 0, y'_2 \mapsto 0\}$.
- Potential repair candidates: All y_i where $\sigma[y_i] \neq \sigma[y'_i]$.
- $\varphi(X,Y)$ is Boolean Relation.
  - So it can be $\hat{\sigma} = \{x_1 \mapsto 1, x_2 \mapsto 1, y_1 \mapsto 0, y_2 \mapsto 1, y'_1 \mapsto 0, y'_2 \mapsto 0\}$
  - We would not repair f_1.
- MaxSAT-based Identification of *nice counterexamples*:
  - Hard Clauses $\varphi(X,Y) \land (X \leftrightarrow \sigma[X])$.
  - Soft Clauses $(Y \leftrightarrow \sigma[Y'])$.
- Candidates to repair: Y variables in the violated soft clauses

Repairing Approximations

- $\sigma = \{x_1 \mapsto 1, x_2 \mapsto 1, y_1 \mapsto 0, y_2 \mapsto 1, y_1' \mapsto 0, y_2' \mapsto 0\}$, and we want to repair f_2.

- Potential Repair: If $\underbrace{x_1 \wedge x_2 \wedge \neg y_1}_{\beta=\{x_1, x_2, \neg y_1\}}$ then $y_2 = 1$

- Would be nice to have $\beta = \{x_1, x_2\}$ or even $\beta = \{x_1\}$

- Challenge: How do we find small β?
  - $G_\sigma(X, Y) := \varphi(X, Y) \wedge x_1 \wedge x_2 \wedge \neg y_1 \wedge \neg y_2$
  - $\beta :=$ Literals in UNSAT Core of $G_\sigma(X, Y)$

Repair: Adding Level to Decision List

- Candidates are from one level decision list:
  - Say we have paths p_1, p_2 with the leaf node label as 1.
  - Learned decision tree: If p_1 then 1, elif p_2 then 1, else 0.
  - p_1, p_2 can be reordered.

Repair: Adding Level to Decision List

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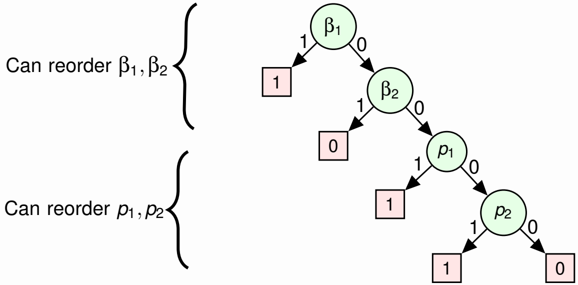
  - Learned decision tree: If p_1 then 1, elif p_2 then 1, else 0.
  - p_1, p_2 can be reordered.
- Suppose in repair iterations, we have learned: If β_1 then 1, $\ldots \beta_2$ then 0

  ……..
- β_1 and β_2 can be reordered.
- From one-level decision list to two-level decision list.



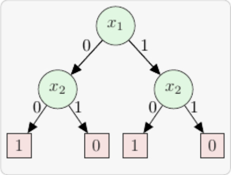
Manthan

$\varphi(X, Y)$
$X = \{x_1, x_2\}$
$Y = \{y_1, y_2\}$

Data Generation →

x_1	x_2	y_1	y_2
0	0	1	0
0	1	0	1
1	0	1	0
1	1	0	1

Learn Candidates →



Verify Candidates ↓

Check Satisfiability of $E(X, Y, Y')$

SAT, σ → $G_\sigma(X, Y)$

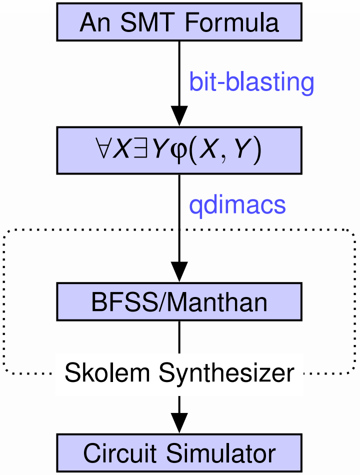
UNSAT Core-based Repair

UNSAT ↓

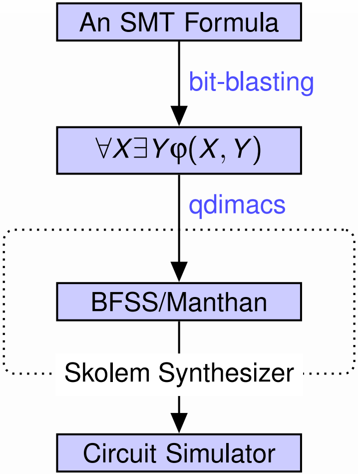
Return F

Outline

Tool Demo: Pipeline



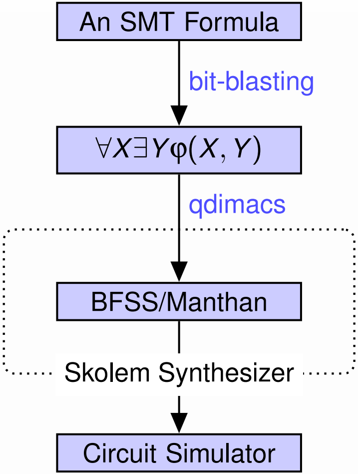
Tool Demo: Pipeline




```
(set-logic BV)
;; out function with two 2 bit arguments
(declare-fun out ( (_ BitVec 2) (_ BitVec 2)) (_ BitVec 2))
;; declaring the constant
(declare-const inp1 (_ BitVec 2))
(declare-const inp2 (_ BitVec 2))
;; output of out function should be greater than or equal to first input
(assert (bvuge (out inp1 inp2) inp1))
;; output of out function should be greater than or equal to second input
(assert (bvuge (out inp1 inp2) inp2))
;; output of out function should be either be equal to first input
;; or to the second input
(assert (or (= inp1 (out inp1 inp2)) (= inp2 (out inp1 inp2)) (= inp3 (out inp1 inp2) )))
(check-sat)
```

An SMT formula

Tool Demo: Pipeline

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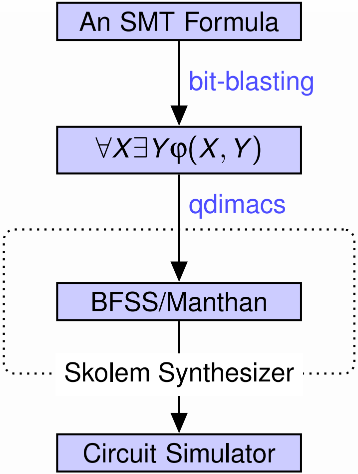
```

An SMT formula

```
p cnf 12 32
a 3 5 7 8 0
e 1 2 4 6 9 10 11 12 0
1 -2 0
3 1 0
2 -3 -1 0
4 -5 0
1 -5 0
4 1 0
-2 6 0
7 6 0
2 -7 -6 0
4 -8 0
-8 6 0
4 6 0
-4 -8 -9 0
```

Qdimacs formula

Tool Demo: Pipeline

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4 1 0
-2 6 0
7 6 0
2 -7 -6 0
4 -8 0
-8 6 0
4 6 0
-4 -8 -9 0
```
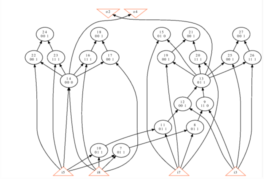

Qdimacs formula



Synthesized Skolem function

Outline

Summary

- Functional Synthesis is a fundamental problem with wide variety of applications
  - program synthesis, games and planning, circuit repair
- Long history of work that has sought to push the scalability envlope
- An exciting and diverse set of approaches
  - Knowledge compilation
  - Guess, check, and repair
- Promise of scalability: Out of 609 benchmarks

2018	247 solved
2019	280 solved
2020	356 solved
2021	509 solved

Where do we go from here?

1. Benchmarks
2. Notion of Quality
3. Beyond Single Functions
4. Beyond Propositional Logic

Future Directions I: Benchmarks

Promise of scalability: Out of 609 benchmarks

- 2018 SOTA 247 solved
- 2019 SOTA 280 solved
- 2020 SOTA 356 solved
- 2021 SOTA 509 solved

B. Cook, 2022: **Virtuous cycle in Automated Reasoning**: ...application areas drives more investment in foundational tools, while improvements in the foundational tools drive further applications. Around and around.

Future Directions II: Search for Optimal Functions

- The current formulation allows the solver to find an arbitrary functions
- Opportunity to formalize the notion of quality
- Smaller size?
- Uses gates of particular type?

Future Directions III: Beyond Single Functions

- Enumeration of functions: Knowledge compilation
- Uniform sampling of functions: randomized strategies
- Counting of functions

Future Directions IV: Beyond Propositional Logic

- Past twenty years: Development of solvers with satisfiability modulo theory solvers
  - Capable of handling theories such as string, bitvectors, linear real arithmetic
- Lifting synthesis techniques to SMT
  - Knowledge compilation
  - Machine Learning techniques for SMT learning
  - Repair techniques

Additional Slides

A Quick Aside

Many questions required solving QBF.

But how do we go from QBF to 2-QBF?

Application Domain 2: Games and planning from QBF to 2-QBF/dep-QBF

A Quick Aside

Many questions required solving QBF.

But how do we go from QBF to 2-QBF?

Two simple ways

1. QBF to 2-QBF by repeated substitutions of Skolem functions!

1. Remove inner most quantifier alternation.
2. Substitute Skolem function.

- E.g., if $\Psi = \forall \mathbf{X}_1 \exists \mathbf{Y}_1 \forall \mathbf{X}_2 \exists \mathbf{Y}_2 \varphi$, consider 2-QBF formula $\Psi' = \forall \mathbf{X}_1 \forall \mathbf{Y}_1 \forall \mathbf{X}_2 \exists \mathbf{Y}_2 \varphi$
- Synthesize Skolem fns F for $\mathbf{Y}_2$ in terms of $\mathbf{X}_1, \mathbf{Y}_1, \mathbf{X}_2$. Let $\varphi_1 = \varphi[\mathbf{Y}_2 \mapsto F]$.

1. QBF to 2-QBF by repeated substitutions of Skolem functions!

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- Synthesize Skolem fns F for $\mathbf{Y}_2$ in terms of $\mathbf{X}_1, \mathbf{Y}_1, \mathbf{X}_2$. Let $\varphi_1 = \varphi[\mathbf{Y}_2 \mapsto F]$.
- Observe: $\Psi \equiv \forall \mathbf{X}_1 \exists \mathbf{Y}_1 \forall \mathbf{X}_2 \varphi_1$

1. QBF to 2-QBF by repeated substitutions of Skolem functions!

1. Remove inner most quantifier alternation.
2. Substitute Skolem function.
3. Use Double Negation

- E.g., if $\Psi = \forall \mathbf{X}_1 \exists \mathbf{Y}_1 \forall \mathbf{X}_2 \exists \mathbf{Y}_2 \varphi$, consider 2-QBF formula $\Psi' = \forall \mathbf{X}_1 \forall \mathbf{Y}_1 \forall \mathbf{X}_2 \exists \mathbf{Y}_2 \varphi$
- Synthesize Skolem fns F for $\mathbf{Y}_2$ in terms of $\mathbf{X}_1, \mathbf{Y}_1, \mathbf{X}_2$. Let $\varphi_1 = \varphi[\mathbf{Y}_2 \mapsto F]$.
- Observe: $\Psi \equiv \forall \mathbf{X}_1 \exists \mathbf{Y}_1 \forall \mathbf{X}_2 \varphi_1 \equiv \exists \mathbf{X}_1 \forall \mathbf{Y}_1 \exists \mathbf{X}_2 \neg \varphi_1$

Application Domain 2: Games and planning from QBF to 2-QBF/dep-QBF

1. QBF to 2-QBF by repeated substitutions of Skolem functions!

1. Remove inner most quantifier alternation.
2. Substitute Skolem function.
3. Use Double Negation
4. Continue

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- Synthesize Skolem fns G for $\mathbf{X}_2$, let $\varphi_2 = \neg \varphi_1[\mathbf{X}_2 \mapsto G]$.

1. QBF to 2-QBF by repeated substitutions of Skolem functions!

1. Remove inner most quantifier alternation.
2. Substitute Skolem function.
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4. Continue

- E.g., if $\Psi = \forall \mathbf{X}_1 \exists \mathbf{Y}_1 \forall \mathbf{X}_2 \exists \mathbf{Y}_2 \varphi$, consider 2-QBF formula $\Psi' = \forall \mathbf{X}_1 \forall \mathbf{Y}_1 \forall \mathbf{X}_2 \exists \mathbf{Y}_2 \varphi$
- Synthesize Skolem fns F for $\mathbf{Y}_2$ in terms of $\mathbf{X}_1, \mathbf{Y}_1, \mathbf{X}_2$. Let $\varphi_1 = \varphi[\mathbf{Y}_2 \mapsto F]$.
- Observe: $\Psi \equiv \forall \mathbf{X}_1 \exists \mathbf{Y}_1 \forall \mathbf{X}_2 \varphi_1 \equiv \exists \mathbf{X}_1 \forall \mathbf{Y}_1 \exists \mathbf{X}_2 \neg \varphi_1$
- Synthesize Skolem fns G for $\mathbf{X}_2$, let $\varphi_2 = \neg \varphi_1[\mathbf{X}_2 \mapsto G]$.
- Then: $\Psi \equiv \exists \mathbf{X}_1 \forall \mathbf{Y}_1 \varphi_2 \equiv \forall \mathbf{X}_1 \exists \mathbf{Y}_1 \neg \varphi_2$ which is in 2-QBF

Application Domain 2: Games and planning from QBF to 2-QBF/dep-QBF

1. QBF to 2-QBF by repeated substitutions of Skolem functions!

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4. Continue

- E.g., if $\Psi = \forall \mathbf{X}_1 \exists \mathbf{Y}_1 \forall \mathbf{X}_2 \exists \mathbf{Y}_2 \varphi$, consider 2-QBF formula $\Psi' = \forall \mathbf{X}_1 \forall \mathbf{Y}_1 \forall \mathbf{X}_2 \exists \mathbf{Y}_2 \varphi$
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- Observe: $\Psi \equiv \forall \mathbf{X}_1 \exists \mathbf{Y}_1 \forall \mathbf{X}_2 \varphi_1 \equiv \exists \mathbf{X}_1 \forall \mathbf{Y}_1 \exists \mathbf{X}_2 \neg \varphi_1$
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- Then: $\Psi \equiv \exists \mathbf{X}_1 \forall \mathbf{Y}_1 \varphi_2 \equiv \forall \mathbf{X}_1 \exists \mathbf{Y}_1 \neg \varphi_2$ which is in 2-QBF

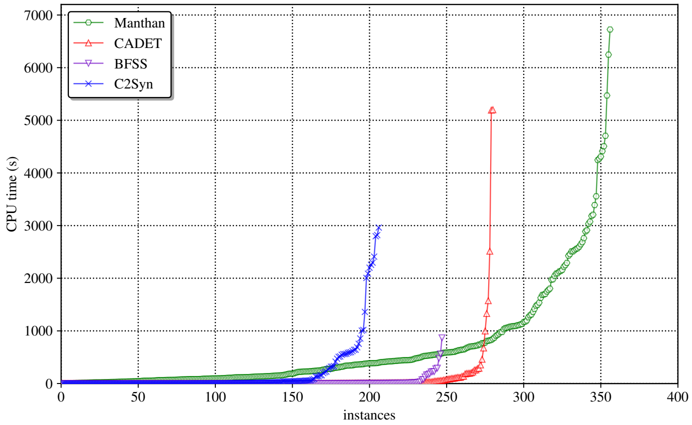
2. QBF to Dep-QBF by exploiting dependencies!

Every QBF formula is equivalent to a (2-)dep-QBF formula!
E.g., $\forall \mathbf{X}_1 \exists \mathbf{Y}_1 \forall \mathbf{X}_2 \exists \mathbf{Y}_2 \forall \mathbf{X}_3 \exists \mathbf{Y}_3 \varphi \equiv \forall \mathbf{X}_1 \forall \mathbf{X}_2 \forall \mathbf{X}_3 \exists^{\{x_1\}} \mathbf{Y}_1 \exists^{\{x_1, y_1, x_2\}} \mathbf{Y}_2 \exists^{\{x_1, y_1, x_2, y_2, x_3\}} \mathbf{Y}_3$.

Experimental Evaluations

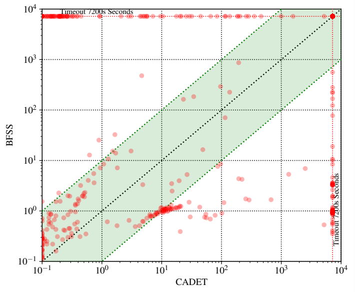
- 609 Benchmarks from:
  - QBFEval competition (`http://www.qbflib.org/`)
  - Arithmetic functions (Tabajara, Vardi,'2017)
  - Disjunctive decomposition (Akshay et al. '2017)
  - Factorization(Akshay et al. '2017)
- Compared among different state-of-the-art tools:
  - CADET (Rabe et al.'2019)
  - C2Syn (Chakraborty et al.' 2019)
  - BFSS (Akshay et al. '2018)
  - Manthan (Golia et al.' '2020,'2021).
- Timeout: 7200 seconds.

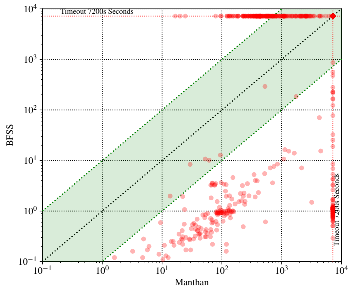
Experimental Evaluations: SOTA'20

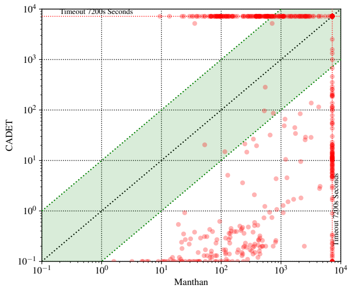


C2Syn	BFSS	CADET	Manthan
206	247	280	356

Experimental Evaluations: SOTA'20





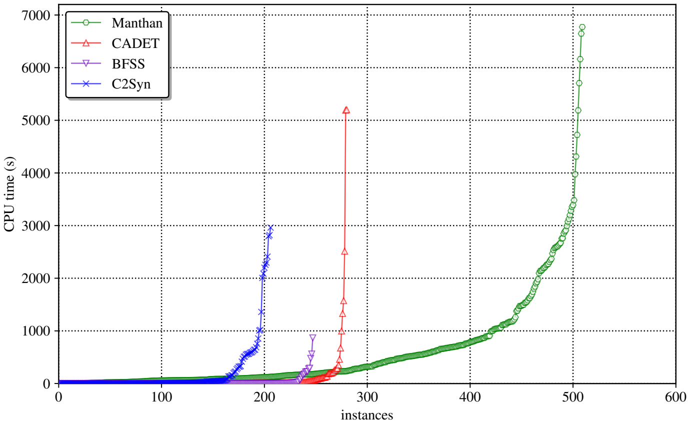


- BFSS \ CADET = 67
- CADET \ BFSS = 100

- BFSS \ Manthan = 85
- Manthan \ BFSS = 194

- CADET \ Manthan = 111
- Manthan \ CADET = 187

Experimental Evaluations: SOTA'21



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206	247	280	509