Quantified Boolean Formulas (QBF)

Quantifier : $\forall$ for all

$\exists$ there exists

$F = x_1 \vee x_2 \vee x_3$

$\phi = \exists x_1, x_2, x_3 \; F$

$\phi = \exists x_1, x_2, x_3 \; (x_1 \vee x_2 \vee x_3)$

True : $\langle x_1 \mapsto 1, x_2 \mapsto 0, x_3 \mapsto 1 \rangle$

ϕ is true.

$$\phi = \forall x_1, x_2, x_3 \; (x_1 \vee x_2 \vee x_3) \longrightarrow \text{False.}$$

$$\phi = \forall x_1 \; \exists x_2, x_3 \; (x_1 \vee x_2 \vee x_3)$$

both   at least one

$\cancel{x_1}$	x_2	x_3
0	1	1
1	1	1

$$\phi = \forall x_1, x_2 \; \exists x_3 \; (x_1 \vee x_2 \vee x_3)$$

$$\phi = \forall x_1 \; \exists x_2 \; \forall x_3 \; (x_1 \vee x_2 \vee x_3)$$

$$\phi = \forall x_1, x_2, x_3 \quad (x_1 \vee x_2 \vee x_3)$$

Is it true ?

No ! $\underline{\langle x_1 \mapsto 0,\ x_2 \mapsto 0,\ x_3 \mapsto 0 \rangle}$ doesn't satisfy $(x_1 \vee x_2 \vee x_3)$

$$\phi = \forall x_1, x_2\ \exists x_3 \quad (x_1 \vee x_2 \vee x_3)$$

x_1	x_2	x_3
0	0	1
0	1	1
1	0	1
1	1	1

$\underline{\phi \text{ is true}}$

$$\phi = \forall x_1 \ \exists x_2 \ \forall x_3 \ (x_1 \vee x_2 \vee x_3)$$

x_1	x_2	x_3
		0
$\underline{\underline{0}}$	1	1
		0
$\underline{\underline{1}}$	0	1

$x_1 \vee x_2 \vee x_3$ is $\underline{\underline{\text{true}}}$

Prenex Normal Form (PNF)

$$\phi = Q_1 x_1 \; Q_2 x_2 \; Q_3 x_3 \; \cdots \; Q_n x_n \; F$$

$\mathrm{var}(F) = \{x_1 \ldots x_n\}$

$$Q \in \{\exists, \forall\}$$

QBF Game Semantics

$\forall$: Universal player $(U_1 \dots U_n)$

$\exists$: Existential player $(E_1 \dots E_n)$

$\forall U_1 \exists E_1 \dots \forall U_n \exists E_n \; F$

→ For all strategies of Universal player, does there exists a strategy of existential player, such that F is true.

- → Φ is true, then existential player wins
- → Φ is false, then universal player wins.

$$\phi = Q_1 x_1 \; Q_2 x_2 \; Q_3 x_3 \cdots Q_n x_n \; F$$

$var(F) = \{x_1 \ldots x_n\}$

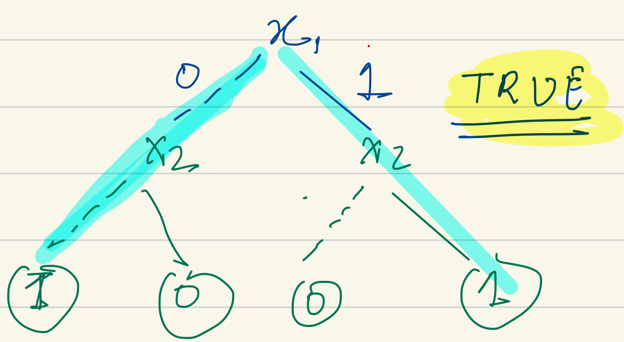
$$Q \in \{\exists, \forall\}$$

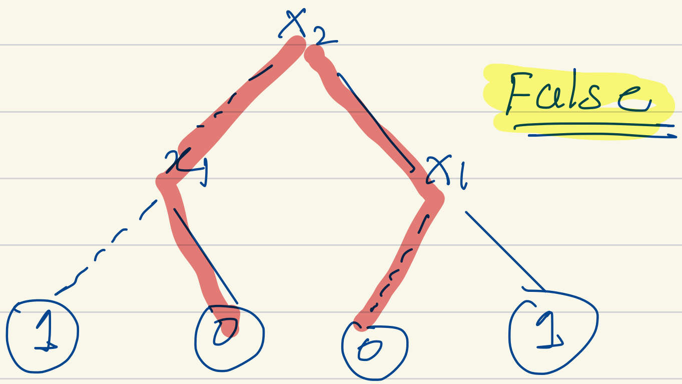
$$\forall x F \equiv F[x \mapsto 1] \wedge F[x \mapsto 0]$$

$$\exists x F \equiv F[x \mapsto 1] \vee F[x \mapsto 0]$$

$\forall x_1 \exists x_2 \; (x_1 \vee \neg x_2) \wedge (\neg x_1 \vee x_2)$

$\exists x_2 \forall x_1 \; (x_1 \vee \neg x_2) \wedge (\neg x_1 \vee x_2)$





QBF

$$\phi = \forall x_1\, x_2\ \exists x_3 \quad x_1 \lor x_2 \lor x_3$$

$$\Rightarrow \quad \forall x_1\, x_2 \quad (x_1 \lor x_2 \lor 1) \lor (x_1 \lor x_2 \lor 0)$$

$$\Rightarrow \forall x_1 \quad ((x_1 \lor 0 \lor 1) \land (x_1 \lor 1 \lor 1)) \lor ((x_1 \lor 1 \lor 0) \land (x_1 \lor 0 \lor 0))$$

$$\Rightarrow (((1 \lor 0 \lor 1) \land (0 \lor 0 \lor 1)) \land ((1 \lor 1 \lor 1) \land (0 \lor 1 \lor 1))) \lor (((1 \lor 1 \lor 0) \land (0 \lor 1 \lor 0)) \land ((1 \lor 0 \lor 0) \land (0 \lor 0 \lor 0)))$$

$$\Rightarrow \quad \underline{1} \lor 0$$

$$\Rightarrow \quad \underline{1} \quad (\text{TRUE})$$

$$\phi = Q_1 x_1 \ \ldots \ Q_n x_n \ F$$

$$Q = \{Q_1 \ \ldots \ Q_n\} \quad , \quad F$$

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QBF solver (Q, F):
{
    if Q == [ ]:
            return simplify(F)

  elif Q_1 == ∃ :
          return QBFsolver (Q[1:], F[x_1 ↦ 1]) ∨
                    QBFsolver (Q[1:], F[x_1 ↦ 0])
  else
          return QBFsolver (Q[1:], F[x_1 ↦ 1]) ∧
                    QBFsolver (Q[1:], F[x_1 ↦ 0])
}
```

Optimizations :-

Unit clause

C is called unit clause in formula Φ iff

- C contains only one existential literal
- Universal literals of C are to the right of the existential literals in C.

Unit literal : existential literal in Unit clause.

Unit literal Elimination

$x_a \vee x_2$

l be unit literal in ϕ

- → removing all clauses containing l
- → removing all occurrences of $\neg l$.

Identify unit clause, unit literal & perform unit literal elimination (if possiable)

1. $\forall x_1 \; \exists x_2. \; (\underline{x_1} \vee \neg x_2) \wedge (\neg \underline{x_1} \vee x_2)$

2. $\exists x_2 \; \forall x_1 \; (x_1 \vee \neg x_2) \wedge (\neg x_1 \vee x_2)$

Optimizations :-

$$\phi = \exists x\ F$$

$$\simeq \quad F(x \mapsto 0) \vee F(x \mapsto 1)$$

what happens when "x" occurs only positively in the formula?

$$\exists x\ F \simeq F(x \mapsto 1)$$

&

$$\exists x\ F \simeq F(x \mapsto 0)$$ if "x" occurs only negatively in the formula.

$$\phi = \forall x \; F$$

$$\hat{=} \quad F(x \mapsto 1) \wedge F(x \mapsto 0)$$

x occurs only positively in the formula

$$\forall x F \hat{=} F(x \mapsto 0)$$

x occurs only negatively in the formula

$$\forall x F \hat{=} F(x \mapsto 1)$$

Pure literal Elimination

if l is pure literal

- → remove all clauses with l, if l is existentially quantified.
- → remove all occurences of l if l is universally quantified.

Q- Resolution = Resolution + $\forall$ Reduction

- Resolution :

$$\frac{l \vee \alpha \qquad \neg l \vee \beta}{\alpha \vee \beta}$$

→ l is existentially quantified.

- $\forall$- reduction (Universal Reduction).

$$\frac{\alpha \vee l}{\alpha}$$, where l is universally quantified variables, & all other variables of α are at left of l in prenex.

Perform Q-Resolution on:

1. $\forall x_1 \; \exists x_2 \quad (x_1 \vee \neg x_2) \wedge (x_1 \vee x_2)$

2. $\forall x_1 \; \exists x_2 \; (x_1 \vee \neg x_2) \wedge (\neg x_1 \vee x_2)$

$\forall x_1 \; \exists x_2 \; (x_1 \lor \neg x_2) \land (x_1 \lor x_2)$

$(x_1 \lor \neg x_2) \quad \land \quad (x_1 \lor x_2)$

(Resolve on $\exists x_2$)

x_1 ($\forall$-reduction of x_1)

[ ]

$\perp$

$$\forall x_1 \; \exists x_2 \; (x_1 \vee \neg x_2) \wedge (\neg x_1 \vee x_2)$$

← $\exists x_2$ Resolution

$$(x_1 \vee \neg x_1)$$

← $\forall$ Reduction

$$\perp$$

unsound.

Tautologous resolvents are generally unsound. & not allowed.

Q- Resolution = Resolution + ∀ Reduction

- Resolution :

$$\frac{l \vee \alpha \qquad \neg l \vee \beta}{\alpha \vee \beta}$$

→ l is existentially quantified.

- ∀- reduction (Universal Reduction).

$$\frac{\alpha \vee l}{\alpha}$$, where l is universally quantified variables, & all other variables of α are at left of l in prenex.
only if $\alpha \vee l$ is not tautology.

Come up QDPLL algorithm using:

1. Basic Algorithm & Q-Resolution & optimizations!

Thanks :)